Data-Driven Auction Design III

Learnability of Product Distributions and Strong Revenue Monotonicity

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Recap: Single-Item Auctions

- Sell 1 item to n bidders, to maximize revenue
- Bidder i's value v_i is drawn independently from D_i

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  1. Bidders bid $b_1, b_2, \ldots, b_n$
  2. Seller picks allocations $x_1, x_2, \ldots, x_n$ and payments $p_1, p_2, \ldots, p_n$
  3. Bidder i wins the item w.p. x_i, pays p_i, gets utility $v_i x_i - p_i$

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- Dominant-Strategy Incentive Compatible (DSIC)

$$\forall i, v_i, b_i, b_{-i} : \quad v_i x_i(v_i, b_{-i}) - p_i(v_i, b_{-i}) \geq v_i x_i(b_i, b_{-i}) - p_i(b_i, b_{-i})$$

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- Individually Rational (IR)

$$\forall i, v_i, b_{-i} : \quad v_i x_i(v_i, b_{-i}) - p_i(v_i, b_{-i}) \geq 0$$

Recap: Myerson's Theory

- DSIC and IR are equivalent to
  1. $x_i(v_i, b_{-i})$ is monotone (e.g., step function)
  2. $p_i(v_i, b_{-i})$ is the area on the left of $x_i(v_i, b_{-i})$ as a function of v_i (e.g., threshold price above which $x_i = 1$, if x_i is a step function)

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- Expected revenue is equivalent to expected virtual welfare

$$\mathbf{E} \sum_{i=1}^{n} \varphi_i(v_i) x_i$$

where the virtual value φ_i is

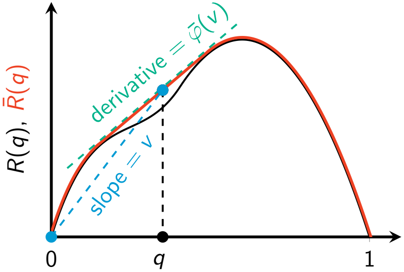
$$\varphi_i(v_i) = v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}$$

Recap: Myerson's Optimal (Single-Item) Auction

- $\bar{R}(q)$ is concave closure of revenue curve
  - Max expected revenue given sale prob. q
- Ironed virtual value $\bar{\varphi}_i(v_i)$ is $\bar{R}(q)$'s derivative
  - Quantile q's marginal revenue contribution

- Highest non-negative ironed virtual value wins
- Winner pays threshold winning bid
  i.e., lowest bid above which he/she wins

- Expected revenue is at most $\mathbf{E} \sum_{i=1}^{n} \bar{\varphi}_i(v_i) x_i$
  with equality if values in an ironed interval are treated as the same



Recap: Data-Driven Optimal (Single-Item) Auction

- Sample Complexity/Statistical Learning Model
  - Take m i.i.d. samples from $D = D_1 \times D_2 \times \cdots \times D_m$ as input
  - Output a DSIC and IR auction A
- How many samples are needed to pick a near optimal A "up to an ε margin"?
  - ε additive approximation

    $[0, 1]$-bounded distributions **(illustrative example)**
  - $1 - \varepsilon$ (multiplicative) approximation

    Regular distributions (i.e., concave revenue curve)

    MHR distributions (i.e., "strongly concave" revenue curve)

    $[1, H]$-bounded distributions
- The sample complexity is smallest number of samples needed

Recap: Summary of Upper and Lower Bounds So Far

Distributions	Upper Bound	Lower Bound
$[0, 1]$-Bounded	$\frac{n}{\varepsilon^3}$	$\frac{n}{\varepsilon^2}$
Regular distributions	$\frac{n}{\varepsilon^4}$	$\frac{n}{\varepsilon^3}$
MHR distributions	$\frac{n}{\varepsilon^3}$	$\frac{n}{\varepsilon^2}$
$[1, H]$-bounded distributions	$\frac{Hn}{\varepsilon^3}$	$\frac{Hn}{\varepsilon^2}$

- **Upper Bound:**
  Concentration inequality + covering of auction space + union bound
- **Lower Bound:**
  Assouad's method

Recap: Concentration Inequalities

Theorem (Chernoff-Hoeffding, User-Friendly Version)

$X_1, X_2, \ldots, X_m$ are i.i.d. RV over $[0,1]$. Let $\mu = \mathbf{E}\, X_i$. With probability $1-\delta$ we have

$$\left|\frac{1}{m}\sum_{i=1}^{m} X_i - \mu\right| \lesssim \sqrt{\frac{\log\frac{1}{\delta}}{m}}$$

Theorem (Bernstein Inequality, User-Friendly Version)

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$$\left|\frac{1}{m}\sum_{i=1}^{m} X_i - \mu\right| \lesssim \max\left\{\sqrt{\frac{\mu(1-\mu)\log\frac{1}{\delta}}{m}}, \frac{\log\frac{1}{\delta}}{m}\right\}$$

Learning Prices' Revenue vs. Learning Value Distribution

- Recall our approach for data-driven optimal pricing
  - It suffices to learn the revenue of every price up to ε
  - For each price, estimating its revenue reduces to estimating its quantile

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  - It suffices to learn the distribution up to ε w.r.t. its CDF/quantile, i.e.

$$\sup_{v \in [0,1]} \left| \underbrace{F_D(v)}_{\text{true CDF}} - \underbrace{F_E(v)}_{\text{estimated CDF}} \right| \leq \varepsilon \qquad \text{(Kolmogorov distance)}$$

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(simplified incorrect form for illustration)

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- Return Myerson's optimal auction w.r.t. E or $\bar{E}$

Hellinger Distance

$$\mathrm{H}(P, Q) = \frac{1}{\sqrt{2}} \left\| \sqrt{P} - \sqrt{Q} \right\|_2 = \sqrt{\frac{1}{2} \sum_{v} \left(\sqrt{P(v)} - \sqrt{Q(v)} \right)^2}$$

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$$1 - \mathrm{H}(P_1 \times \cdots \times P_n, Q_1 \times \cdots \times Q_n)^2 \; = \; \prod_{i=1}^{n} \left(1 - \mathrm{H}(P_i, Q_i)^2 \right)$$

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- This implies sub-additivity

$$\mathrm{H}(P_1 \times \cdots \times P_n, Q_1 \times \cdots \times Q_n)^2 \;\leq\; \sum_{i=1}^{n} \mathrm{H}(P_i, Q_i)^2$$

Hellinger, Kullback–Leibler, and Total Variation

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$$\mathrm{H}(P, Q)^2 \leq \mathrm{TV}(P, Q) \leq \sqrt{2} \cdot \mathrm{H}(P, Q)$$

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- Why TV is called total variation distance?
  - P and Q are distributions over $\mathcal{T}$
  - $h : \mathcal{T} \to [0, 1]$ is a function
  - We have

$$\left| \mathbf{E}_{v \sim P} h(v) - \mathbf{E}_{v \sim Q} h(v) \right| \leq \mathrm{TV}(P, Q)$$

Learnability of Distribution

Theorem

If D has support size k, E is empirical distribution over $m \precsim \frac{k+\log\frac{1}{\delta}}{\varepsilon^2}$ i.i.d. samples, then

$$\mathrm{H}(D, E) \leq \varepsilon$$

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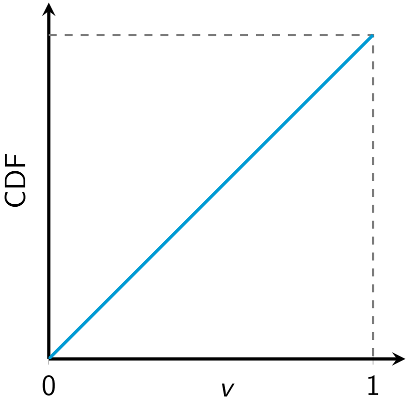
- $E = E_1 \times E_2 \times \cdots \times E_n$ is product empirical distribution from m rounded samples
  - E_i is the uniform distribution over bidder i's rounded sample values

Theorem

With $m \gtrsim \frac{n}{\varepsilon^3} + \frac{\log \frac{1}{\delta}}{\varepsilon^2}$ samples, Myerson's optimal auction M_E w.r.t. E is an ε additiive approximation w.p. $1 - \delta$.

Underestimating Value Distribution

- Value distribution D, e.g., uniform on $[0, 1]$



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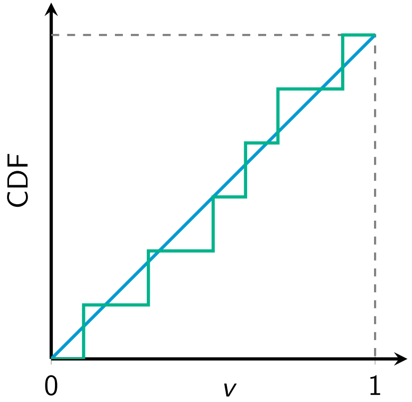

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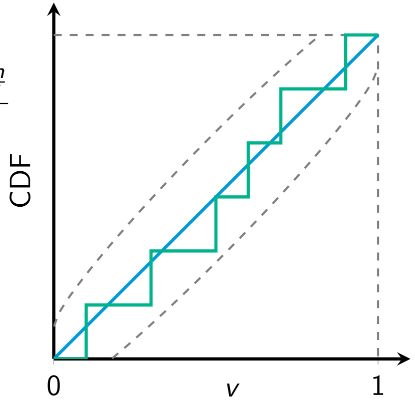

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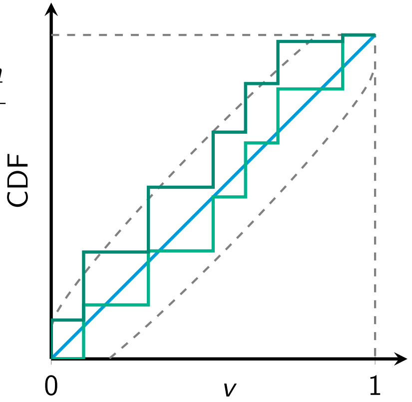

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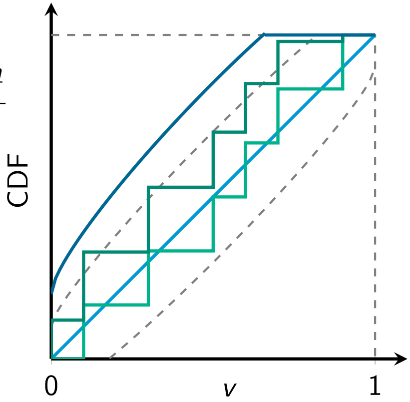

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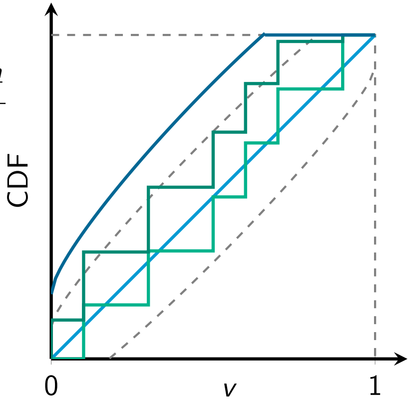

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- Compare $OPT(\bar{D})$ and $OPT(D)$

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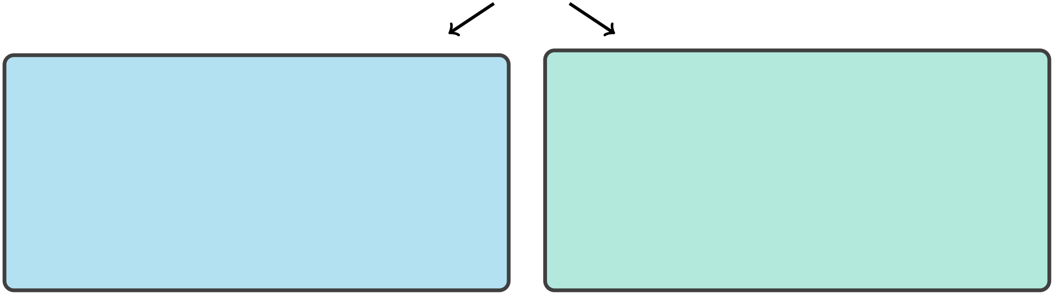

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Comparing $OPT(D)$ and $OPT(\bar{D})$

Reminder

$$F_{\bar{D}}(v) - F_D(v) \approx \sqrt{\frac{F_D(v)(1-F_D(v))\log\frac{m}{\delta}}{m}} + \frac{\log\frac{m}{\delta}}{m}$$

Lemma

If we have $m \gtrsim \frac{n \cdot (\log \frac{m}{\varepsilon\delta})^2}{\varepsilon^2}$ *samples, then the auxiliary distribution* $\bar{D}$

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Summary

Distributions	Sample Complexity
$[0,1]$-Bounded	$\frac{n}{\varepsilon^2}$
Regular distributions	$\frac{n}{\varepsilon^3}$
MHR distributions	$\frac{n}{\varepsilon^2}$
$[1,H]$-bounded distributions	$\frac{Hn}{\varepsilon^2}$

- **Upper Bound:**
  Learnability of product distribution + strong (revenue) monotonicity
- **Lower Bound:**
  Assouad's method

Recap

Two Different Viewpoints

Learnability of Product Distributions

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Further Extensions and Open Questions

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- **Open question:** Is there a Bernstein-style DKW inequality?

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  - **Open question:** Is the slower convergence rate avoidable?

References

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