CSIS0351/CSIS8601: Randomized Algorithms
Lecture 2: Derandomization, More on Probabilistic Method
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1 Derandomization by Conditional Expectation

In the last lecture, we saw randomized algorithms for MAX CUT and MAX 3-SAT. In each of these algorithms, there is an underlying random process involving some random variables $X_0, X_1, \ldots, X_{n-1}$, and we have computed the expectation of some objective value Y, which is a function on those random variables.

We show that under some very general conditions, the randomized algorithm can be *derandomized*, i.e., there is some deterministic algorithm that finds values $X_0 := x_0, X_1 := x_1, \ldots, X_{n-1} := x_{n-1}$ such that the objective value Y is at least its expectation.

1.1 Principle of Conditional Expectation

The success of the derandomization method depends on the following conditions.

Sufficient Conditions

1. The objective value Y is a function of the random variables $X_0, X_1, X_2, \ldots, X_{n-1}$, i.e., if the values of the random variables are known, then the value of Y is uniquely determined.

2. Given a partial assignment[1] $X_{[i]} := x_{[i]}$ for $1 \leq i \leq n$, the conditional expectation $E[Y|X_{[i]} = x_{[i]}]$ can be computed efficiently.

We show that when such conditions are met, it is possible to find values $x_{[n]} := (x_0, x_1, \ldots, x_{n-1})$ for the random variables $X_{[n]} := (X_0, X_1, \ldots, X_{n-1})$ such that the objective value Y is at least its expectation $\mu := E[Y]$.

1. **Initialization.** We begin when none of the random variables has been assigned any values, i.e. $i := 0$. We have the invariant: $E[Y|X_{[i]} = x_{[i]}] \geq \mu$. The left hand side is simply $E[Y]$.

2. **Assigning Value to One More Random Variable X_i.** Suppose for some $0 \leq i < n$, we already have the assignment $X_{[i]} := x_{[i]}$ and the invariant $E[Y|X_{[i]} = x_{[i]}] \geq \mu$. We show the following claim.

   Claim 1.1 *There exists some assignment $X_i := x_i$ such that $E[Y|X_{[i+1]} = x_{[i+1]}] \geq \mu$.*

   Proof: Conditioning on the value of X_i, we have

[1] Recall we denote $[i] := \{0, 1, 2, \ldots, i-1\}$, and $[0] := \emptyset$.

$E[Y|X_{[i]} = x_{[i]}] = \sum_x Pr(X_i = x|X_{[i]} = x_{[i]})E[Y|X_{[i]} = x_{[i]} \wedge X_i = x]$,

where the summation is over the values x that X_i can take. Observe that by the invariant, that the left hand side is at least μ. Hence, it follows that there exists some x_i such that $E[Y|X_{[i]} = x_{[i]} \wedge X_i = x_i] \geq \mu$. ■

For each x, we test if $E[Y|X_{[i]} = x_{[i]} \wedge X_i = x] \geq \mu$ and find such an x_i.

We assign $X_i := x_i$.

3. This continues until $i = n$, when all random variables have received their values. In this case, we have $E[Y|X_{[n]} = x_{[n]}] \geq \mu$, which means that under those values, the objective value Y is at least μ.

1.2 Derandomization for MAX CUT

Given a graph $G = (V, E)$, where $V := [n]$ is labeled by n integers. The randomized algorithm essentially assigns independently for each vertex i, a random variable X_i taking values uniformly in $\{0, 1\}$ (each with probability $\frac{1}{2}$). The cut can be defined by $C := \{i : X_i = 1\}$.

The number Y of edges in the cut is a function of the random variables $X_0, X_1, \ldots, X_{n-1}$. For each edge $e = \{i, j\} \in E$, define Y_e to be 1 if $X_i \neq X_j$ and 0 if $X_i = X_j$. Then, $Y := \sum_{e \in E} Y_e$.

It suffices to show that given a partial assignment $X_{[i]} := x_{[i]}$, the conditional expectation $E[Y|X_{[i]} = x_{[i]}]$ can be computed efficiently. By linearity of expectation, it is enough to consider, for each edge $e = \{u, v\} \in E$, the quantity $E[Y_e|X_{[i]} = x_{[i]}]$. There are 3 cases to consider.

1. If none of X_u or X_v is assigned a value yet, the conditional expectation of Y_e is $\frac{1}{2}$, as before.
2. If exactly one of X_u and X_v is assigned a value, check that the conditional expectation Y_e is also $\frac{1}{2}$.
3. If both of X_u and X_v already have been assigned values, then Y_e is 1 if they receive different values and 0 otherwise.

The running time of the derandomization algorithm is $O(mn)$, where m is the number of edges.

1.3 Derandomization of MAX 3-SAT

The argument is similar. The important part is given a partial assignment of variables, what is the (conditional) probability that a clause is satisfied? There are some cases to consider:

1. If the partial assignment makes the clause satisfied, then it is 1;
2. if there are 3 unassigned variables in the clause, then it is $\frac{7}{8}$;
3. if there are 2 unassigned variables in the clause, then it is $\frac{3}{4}$;
4. if there is 1 unassigned variable in the clause, then it is $\frac{1}{2}$;

5. if there is no more unassigned variables in the clause, then it is 0.

One can check that for m clauses in n variables, the derandomized procedure takes time $O(mn)$.

2 Graphs with No Short Cycles: Method of Alteration

When we use the probabilistic method, after we run the experiment, sometimes we have to make minor alteration to the outcome in order to obtain a desirable solution. We demonstrate this method by considering the number of edges in a graph with no short cycles.

Definition 2.1 *An undirected graph $G = (V, E)$ contains a* cycle *of length l if there are l vertices $v_1, v_2, \ldots, v_l$ such that all l edges $\{v_1, v_2\}, \{v_2, v_3\}, \ldots \{v_{l-1}, v_l\}, \{v_l, v_1\} \in E$ are present. The minimum length of a cycle is 3; note that there is no cycle of length 2.*

Question. Suppose a graph has no cycles of length l or less. What is the maximum number of edges that it can have?

Observe that we are trying to optimize two conflicting objectives: adding more edges means eventually creating short cycles. In the extreme case, in a complete graph, every 3 points form a 3-cycle.

Theorem 2.2 *There exists an n-vertex graph with no cycles of length l or less that has at least $\Omega(n^{1+\frac{1}{l-1}})$ edges.*

We proof the special case for $l = 3$. The general case will appear in a homework question.

Definition 2.3 (Random Graph $G_{n,p}$) *Consider the following experiment. Let V be a set of n vertices. We form a random graph (V, E) in the following way. For each unordered pair $\{u, v\} \in \binom{V}{2}$, independently add an edge between u and v with probability p, i.e., $Pr(\{u, v\} \in E) = p$. The resulting graph is known as $G_{n,p} := (V, E)$.*

Note that for any graph G with index set V, $Pr(G_{n,p} = G) > 0$. Our candidate graph could be generated by the process. If we want more edges, then p should be large; if we do not want short cycles, then p should be small. We will find the best value of p to balance between the two requirements..

2.1 Without alteration: How lucky can we be?

Our best hope is to prove that with non-zero probability, both of the following events A and B happen. Event A is the event that $G_{n,p}$ has a large number X of edges. Event B is the event that there are no triangles in $G_{n,p}$.

Consider event B first. Group the n vertices into $\frac{n}{3}$ groups, each of size 3. (For the time being, assume n is divisible by 3.) Look at one such group. The probability that there is no triangle between 3 vertices is $(1 - p^3)$. The probability that this holds for all $\frac{n}{3}$ groups is $(1 - p^3)^{n/3}$. Hence, $Pr(B) \leq (1 - p^3)^{n/3}$, which is quite small (exponentially small with respect to n).

Note that the events A and B are not independent. If we want to use the union bound $Pr(\overline{A} \cup \overline{B}) \leq Pr(\overline{A}) + Pr(\overline{B})$ to give an upper bound on the failure probability, we would need to prove something like the failure probability $Pr(\overline{A})$ is less than $(1-p^3)^{n/3}$. Since this is small, it is difficult to show that the number of edges X is large using this approach.

Consider another approach. Although with high probability, there would exist a triangle, the number Y of triangles is not too big. Here is an idea. We run the experiment and form $G_{n,p}$. Let X be the number of edges and Y be the number of triangles. We pick one edge from each triangle and remove it. In the worst case, we remove Y edges from the graph.

After this alteration, the graph would have no triangles, and the number of edges in the remaining graph is at least $Z := X - Y$. By choosing the probability p carefully, we show that with non-zero probabilitiy, Z is large.

First observe the following quantities.

Claim 2.4 *We have the following.*

1. $E[X] = \binom{n}{2}p$
2. $var[X] = \binom{n}{2}p(1-p)$
3. $E[Y] = \binom{n}{3}p^3$.

Proof: The first two results follow from the fact that X has a binomial distribution with $\binom{n}{2}$ objects and probability p. The last result follows from the fact that there are $\binom{n}{3}$ ways to form a cycle in a graph, and the probability that each of them is formed is p^3. ■

We can proceed in two ways: (1) using probability or (2) using expectation.

2.2 By Probability

Suppose we can show the following:

1. For some $\alpha > 0$, $Pr(X < E[X] - \alpha) < \frac{1}{2}$.
2. For some $\beta > 0$, $Pr(Y > \beta) < \frac{1}{2}$.

Using union bound, we know that with non-zero probability neither events happen, and in that case, we have $Z := X - Y \geq E[X] - \alpha - \beta$.

We first choose p such that $E[X] \geq 4E[Y]$. Observe that we can choose $p := \sqrt{\frac{3}{8n}}$. Check that $E[X] = \Theta(n^{1.5})$.

For the first event, observe by Chebyshev's inequality, $Pr(X < E[X] - \alpha) \leq Pr(|X - E[X]| > \alpha) < \frac{var[X]}{\alpha^2} \leq \frac{\binom{n}{2}p}{\alpha^2}$. The last quantity is at most $\frac{1}{2}$, if we set $\alpha := \sqrt{2\binom{n}{2}p} = \Theta(n^{0.75})$.

For the second event, by Markov's inequality, if we set $\beta := 2E[Y]$, then $Pr[Y > \beta] < \frac{1}{2}$.

Hence, it follows that with non-zero probability, we have

$Z := X - Y \geq E[X] - \alpha - 2E[Y] \geq E[X] - \alpha - \frac{1}{2}E[X] = \frac{E[X]}{2} - \sqrt{2\binom{n}{2}p} \geq \Omega(n^{1.5})$.

2.3 By Expectation

We choose p such that $E[X] \geq 2E[Y]$. We can set $p := \sqrt{\frac{3}{4n}}$. Check that $E[X] = \Theta(n^{1.5})$.

Then, it follows that

$E[Z] = E[X] - E[Y] \geq E[X] - \frac{E[X]}{2} = \frac{E[X]}{2} = \Omega(n^{1.5})$.

Remark 2.5 Note that this is not the best result for triangle-free graphs. Consider a complete bipartite graph with $\frac{n}{2}$ vertices on each side. Then, the graph has no triangles and has $\Omega(n^2)$ edges. However, we obtain a weaker result using the probabilistic method to illustrate how a similar result could be proved for general l.

2.4 General Case

The general case would appear as a homework problem. If you would like a head start to work on the next homework, here is a preview.

1. **Graphs with No Short-Cycles.** In this question, we show the following result. For each $l \geq 3$, and $n \geq 2^{l+2}$, there exists a graph, with n vertices and no cycles of length l or less, that has $\Omega(n^{1+\frac{1}{l-1}})$ edges.

   (a) Consider the random graph $G_{n,p}$, where $p \geq \frac{2}{n}$. For $3 \leq i \leq l$, let Y_i be the number of length-i cycles in $G_{n,p}$. Compute $E[Y_i]$.

   (b) Let $Y := \sum_{3 \leq i \leq l} Y_i$. Show that $E[Y] \leq (np)^l$.

   (c) By choosing an appropriate value of p, prove that there exists an n-vertex graph, with no cycles of length l or less, that has $\Omega(n^{1+\frac{1}{l-1}})$ edges.

   (d) Derandomize the above procedure, i.e., give a deterministic algorithm that returns a graph with the desired properties. Analyze the running time of your algorithm.