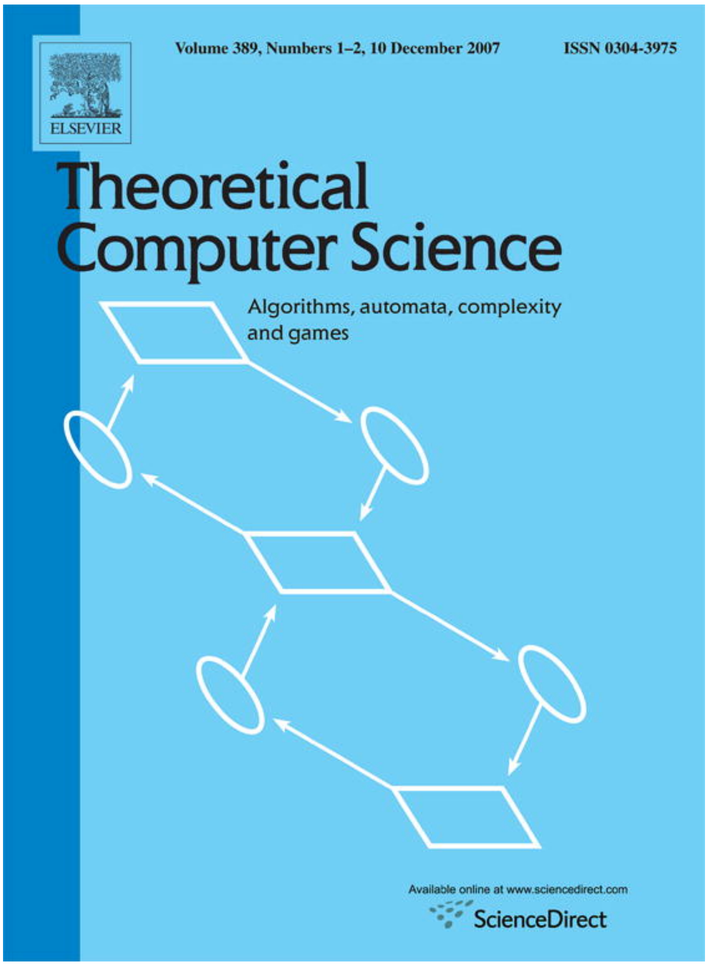
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On the hardness of minimizing space for all-shortest-path interval routing schemes[☆]

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Abstract

k-Interval Routing Scheme (k-IRS) is a compact routing method that allows up to k interval labels to be assigned to an arc; and global k-IRS allows not more than a total of k interval labels in the whole network. A fundamental problem is to characterize the networks that admit k-IRS (or global k-IRS). Many of the problems related to single-shortest-path k-IRS have already been shown to be NP-complete. For all-shortest-path k-IRS, the characterization problem remains open for $k \geqslant 1$. In this paper, we study the time complexity of devising minimal-space all-shortest-path k-IRSs and show that it is NP-complete to decide whether a graph admits an all-shortest-path k-IRS, for every integer $k \geqslant 3$, and so is that of deciding whether a graph admits an all-shortest-path k-strict IRS, for every integer $k \geqslant 4$. These are the first NP-completeness results for all-shortest-path k-IRS where k is a constant and the graph is unweighted. The NP-completeness holds also for the linear case. We also prove that it is NP-complete to decide whether an unweighted graph admits an all-shortest-path IRS with global compactness of at most k, which also holds for the linear and strict cases.

Keywords: Interval routing schemes; Compact routing; NP-completeness

1. Introduction

Interval Routing is a space-efficient routing method for communication networks [13]. The routing table stored at each node groups the set of destination addresses that use the same output port into intervals of consecutive addresses. Formally, the network is modeled as a finite graph $G = (V, E)$, where the set of vertices, V, represents the nodes of the network, and the set of edges, E, represents the bidirectional links. An edge (u, v) between the nodes u and v

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induces two opposite arcs: $\langle u, v\rangle$ and $\langle v, u\rangle$; the $2|E|$ induced arcs form an arc set A of G. A routing scheme for network G assigns each arc $\langle u, v\rangle$ a subset $I(u, v) \subseteq V$, such that the union of the subsets assigned to the arcs emanating from u covers the set $V - \{u\}$. Routing is then performed according to the assignment I, such that at vertex u, a message will be sent on the arc $\langle u, v\rangle$ whose $I(u, v)$ contains the destination of the message. A good interval routing scheme would try to minimize the number of intervals in $I(u, v)$ over all the arcs by selecting a particular address mapping $L : V \to \{1, 2, \ldots, |V|\}$ and an assignment $I : A \to 2^V$. If each $I(u, v)$ contains not more than k intervals under L, the routing scheme, denoted by $R = (L, I)$, is called a *k-Interval Routing Scheme* (k-IRS).

The standard definition of IRS assumes a single routing path between any two nodes, which imposes the outgoing arcs of a node u to be assigned disjoint subsets, *i.e.*, $I(u, v) \cap I(u, w) = \phi$ for $v \neq w$. Clearly, the routing process with such an IRS is deterministic. A more flexible routing scheme, called *multi-path* IRS or *non-deterministic* IRS [14,9], allows multiple arcs of a node to lead to the same destination; the routing process can pick one of these arcs arbitrarily or according to traffic conditions.

We consider two different models of networks. The weighted model associates each edge of the graph with a positive number, to denote the cost of communication for the edge; the unweighted model assumes the cost of every edge to be one unit. The length of a path under either model is the sum of the costs of the edges in the path. In general, routing along shortest paths is desirable. A *shortest path* IRS always induces shortest paths. A *single-shortest-path* IRS offers a *unique* shortest path between any two vertices in the graph. An *all-shortest-path* IRS is a multi-path IRS that gives exactly *all* the shortest paths between any pair of vertices in the graph.

1 and $|V|$ being considered consecutive, the interval $[a, b]$ with $a > b$ denotes the set $\{i \mid a \leqslant i \leqslant |V|\} \cup \{i \mid 1 \leqslant i \leqslant b\}$. An interval $[a, b]$ is linear if $a \leqslant b$, and circular otherwise. An IRS using only linear intervals is a *linear* IRS, LIRS in short. An IRS is *strict*, denoted by SIRS, if every arc $\langle u, v\rangle$ satisfies $u \notin I(u, v)$. An IRS is denoted by SLIRS if it is both strict and linear.

The space efficiency of an IRS is measured by *compactness*. *Edge compactness* is the maximum, over all the arcs $\langle u, v\rangle$, of the number of intervals in $I(u, v)$; *global compactness* is the sum of the interval numbers over all arcs $\langle u, v\rangle$. The characterization of networks that admit a *shortest path* interval routing scheme with edge compactness k (*i.e.*, k-IRS) or global compactness k (global k-IRS for short) is a fundamental question in this field.[1] Successful work has been done for many special classes of graphs, including trees, outerplanar graphs, hypercubes, meshes, r-partite graphs, interval graphs, unit-circular graphs, tori, 2-trees, chordal rings, and general graphs; see [1,8,11–15] for some examples. A summary of these and other results can be found in [9]. For general graphs, existing complexity results for the weighted/unweighted models and various IRS variants are summarized in Table 1, where an entry for any of the single-path IRS or global compactness refers to both the strict and non-strict versions of the problem; an entry for the all-shortest-path IRS of edge compactness refers to both the linear and non-linear versions of the problem; NPC denotes an NP-complete problem.

All the problems related to single-shortest-path IRS are known to be NP-complete. The NP-completeness for the entries of fixed $k \geqslant 3$ in single-path needs an explanation. It follows from combining the result of [6] with that of [10], or with [16]. In [6], Flammini gave a polynomial-time construction of graphs from binary matrices such that there are at most k blocks of consecutive 1's in each column of the matrix under some row permutation if and only if there is a single-shortest-path $(k + 1)$-IRS for the constructed graph; in [10], Goldberg et al. proved that for every constant $k \geqslant 2$, deciding whether a given binary matrix can be row permuted such that each column has at most k blocks of consecutive 1's is NP-complete; in [16], we strengthened the result by showing that the same NP-completeness holds even if the problem is restricted to symmetric matrices.

For the all-shortest-path IRS case, only partial answers (both positive and negative) have been given. 1-SIRS can be reduced to the consecutive ones property of binary matrices, which can be solved in linear time [2]. Flammini et al. in [7] presented characterizations for 1-SLIRS and 1-LIRS. On the negative side, in [5], it was shown that the optimization problem of determining the minimal k such that a given weighted network belongs to the class of all-shortest-path k-IRS is NP-hard. For unweighted networks, the characterization remains open for all-shortest-path IRS of compactness $k \geqslant 1$ [9]. In the context of global compactness, Flammini et al. in [4] derived the NP-completeness for deciding whether there exists an all-shortest-path global k-IRS for a weighted network and an integer k, and this

[1] Note the stress on "shortest path" because if the shortest path requirement is relaxed, every graph supports a single-path 1-IRS and global $(2|E|)$-IRS.

Table 1
Complexity results on characterization of shortest-path IRSs

Paths represented	Compactness measure	Compactness k	Variants	Graph model: Unweighted	Graph model: Weighted
Single	Edge	$k = 1$	IRS	NPC, [3]	NPC, [3]
			LIRS	NPC, [3]	NPC, [3]
		Fixed $k = 2$	IRS	NPC, [6]	NPC, [6]
			LIRS	NPC, [6]	NPC, [6]
		Fixed $k \geqslant 3$	IRS	NPC, [6,10,16]	NPC, [6,10,16]
			LIRS	NPC, [6,10,16]	NPC, [6,10,16]
		General k	IRS	NPC, [6]	NPC, [5]
			LIRS	NPC, [3]	NPC, [3]
	Global	General k	IRS	NPC, [3]	NPC, [5]
			LIRS	NPC, [3]	NPC, [3]
All	Edge	Fixed $k = 1$	IRS	?	?
			SIRS	P, [2,7]	P, [2,7]
		Fixed $k = 2$	IRS	?	?
			SIRS	?	?
		Fixed $k = 3$	IRS	NPC, this paper	NPC, this paper
			SIRS	?	?
		Fixed $k \geqslant 4$	IRS	NPC, this paper	NPC, this paper
			SIRS	NPC, this paper	NPC, this paper
		General k	IRS	NPC, this paper	NPC, [5]
			SIRS	NPC, this paper	NPC, [5]
	Global	General k	IRS	NPC, this paper	NPC, [4]
			LIRS	NPC, this paper	NPC, this paper

result is extended in [5] to single-shortest-path IRS. In terms of global compactness and for unweighted networks, the characterization is open for all-shortest-path IRS.

In this paper, we study the characterization question of all-shortest-path IRS under the unweighted graph model, and with respect to both edge and global compactness. Specifically, we prove that the characterization of networks which admit all-shortest-path k-IRS (linear or non-linear) is NP-complete for every constant $k \geqslant 3$, and the characterization of networks which admit all-shortest-path global k-IRS (or its variants) is NP-complete for general k . These results can be easily generalized to the weighted network model, and hence extend substantially the related NP-completeness results of [4,5].

The rest of the paper is organized as follows. The next section gives some formal definitions of the IRS models and their variants and the characterization problems. The NP-completeness results for the edge compactness and global compactness are respectively presented in Sections 3 and 4. Note that [17] is a preliminary report of the results presented in Section 3. We give some conclusive remarks in the last section.

2. Preliminaries

The graphs we consider are connected, loopless, and do not contain multi-edges. The length of a path in the graph is the sum of the costs of the edges in the path; for the unweighted model, this is equal to the number of edges in the path. For an arc $e = \langle u, v\rangle$, $S(u, v)$ denotes the subset of vertices which can be reached from vertex u through a shortest path starting with e; note that $S(u, v) \neq S(v, u)$. We use $Adj(v)$ to denote the set of neighbours of vertex v in the graph.

We define an Interval Routing Scheme (IRS) as follows.

Definition 1. Let $G = (V, E)$ be a graph that induces arc set A. An IRS on G is a pair $\langle L, I\rangle$ where

(1) L is a one-to-one vertex labeling, $L : V \to \{1, 2, \ldots, |V|\}$;
(2) I is an arc labeling, $I : A \to 2^V$, assigning a subset of V to each arc of A, such that for every vertex $u \in V$, $\bigcup_{(u,v)\in E} I(u, v) \bigcup \{u\} = V$;

(3) for every $x, y \in V$:
   (3.1) there exists a sequence of vertices $x = u_0, u_1, \ldots, u_s = y$ such that for $1 \leqslant i \leqslant s$, $y \in I(u_{i-1}, u_i)$; this sequence is called a routing path induced by $\langle L, I \rangle$;
   (3.2) any routing path induced by $\langle L, I \rangle$ between x and y is a simple path of G, *i.e.*, $u_0, u_1, \ldots, u_s$ are mutually different vertices of V.

To save space in the routing table, an IRS expresses $I(u, v)$, the subset of V assigned to an arc $e = \langle u, v \rangle$, with intervals over $\{1, 2, \ldots, |V|\}$.

Definition 2. An interval of $\{1, 2, \ldots, |V|\}$ is one of the following:

(1) A linear interval $[i, j] = \{i, i+1, \ldots, j\}$, where $i, j \in \{1, 2, \ldots, |V|\}$ and $i \leqslant j$;
(2) a circular interval $[i, j] = \{i, \ldots, |V|, 1, \ldots, j\}$, where $i, j \in \{1, \ldots, |V|\}$ and $i > j$; or
(3) the null interval [ ] which is the empty set ϕ.

For simplicity, we will not always strictly distinguish between a vertex v and its label $L(v)$, and will say that a vertex $v \in V$ is contained in an interval $[i, j]$ if $L(v) \in [i, j]$.

Definition 3. Given $U \subseteq V$ and a labeling L of V, we denote by $N(L, U)$ the minimum number of disjoint intervals such that their union is equal to $\{L(v) \bigm| v \in U\}$.

For example, suppose $V = \{v_1, v_2, \ldots, v_9\}$ and $L(v_i) = i$, then $N(L, \{v_5, v_6, v_7, v_9\}) = 2$, because $\{5, 6, 7, 9\} = [5, 7] \cup [9, 9]$, and $N(L, \{v_1, v_2, v_5, v_6, v_7, v_9\}) = 2$ if the circular interval is allowed, 3 otherwise. Apart from this use of the circular interval in expressing $I(u, v)$ as intervals, $I(u, v)$ itself may or may not be allowed to include the starting vertex u. These restrictions give rise to the following variants of IRS.

Definition 4. Let $R = \langle L, I \rangle$ be an IRS on a graph $G = (V, E)$; we say that R is a

(1) Strict Interval Routing Scheme (SIRS) if for every arc $\langle u, v \rangle \in A$, $u \notin I(u, v)$;
(2) Linear Interval Routing Scheme (LIRS) if for every arc $\langle u, v \rangle \in A$ the intervals representing $I(u, v)$ are restricted to be linear;
(3) Strict Linear Interval Routing Scheme (SLIRS) if it is both an SIRS and an LIRS.

When $u \in I(u, v)$, there is an interval on arc $\langle u, v \rangle$ that contains the starting vertex u. An interval on an outgoing arc of a vertex containing the vertex itself is called a *self-enclosing* interval; otherwise, it is a *strict* interval.

Definition 5. Let $R = (L, I)$ be an IRS (SIRS, LIRS, SLIRS, respectively) on a graph $G = (V, E)$, then

- the (edge) compactness of R is the integer $k = \max\{N(L, I(u, v)), N(L, I(v, u)) \bigm| (u, v) \in E\}$; we denote by (edge) k-IRS (k-SIRS, k-LIRS, k-SLIRS, respectively) every IRS (SIRS, LIRS, SLIRS, respectively) of (edge) compactness not more than k;
- the global compactness of R is the integer $k = N(L) = \sum_{(u,v)\in E}(N(L, I(u, v)) + (L, I(v, u)))$; we denote by global k-IRS (k-SIRS, k-LIRS, k-SLIRS, respectively) every IRS (SIRS, LIRS, SLIRS, respectively) of global compactness not more than k.

For practical reasons, we are interested in designing IRSs that induce only shortest paths.

Definition 6. Let $R = \langle L, I \rangle$ be an IRS (SIRS, LIRS, SLIRS, respectively) on a graph $G = (V, E)$; we say that R is

(1) a single-shortest-path IRS (SIRS, LIRS, SLIRS, respectively) if it induces one and only one of the shortest paths between every pair $x, y \in V$; or
(2) an all-shortest-path IRS (SIRS, LIRS, SLIRS, respectively) if it induces exactly the set of all shortest paths between every pair $x, y \in V$.

By the definitions, for all-shortest-path IRS, the arc labeling I does not have much flexibility in assigning subsets to arcs—$I(u, v)$ is either $S(u, v)$ or $S(u, v) \cup \{u\}$; and for all-shortest-path SIRS, the arc labeling I is identical to S, *i.e.*, $I(u, v) = S(u, v)$ for every arc $\langle u, v \rangle$.

Given a graph G and an integer k, the problems of determining whether G supports an all-shortest-path k-IRS (k-SIRS, k-LIRS, k-SLIRS, global k-IRS, global k-SIRS, global k-LIRS, global k-SLIRS, respectively) are defined as follows.

The all-shortest-path k-IRS (k-SIRS, k-LIRS, k-SLIRS, respectively) problem:
Instance: A graph G, and a positive integer k.
Question: Is there an all-shortest-path k-IRS for G?

The all-shortest-path global k-IRS (k-SIRS, k-LIRS, k-SLIRS, respectively) problem:
Instance: A graph G, and a positive integer k.
Question: Is there an all-shortest-path global k-IRS for G?

Clearly, all of these problems are in the class of NP. In fact, given a graph G, an integer k, a vertex labeling L, and an arc labeling I, it can be verified in polynomial time whether $\langle L, I\rangle$ is an all-shortest-path k-IRS (k-SIRS, k-LIRS, k-SLIRS, global k-IRS, global k-SIRS, global k-LIRS, global k-SLIRS, respectively) for G.

In the following, we prove that all of the above problems are NP-complete; in particular, the all-shortest-path k-IRS and k-LIRS problems are NP-complete for every constant $k \geqslant 3$, and the k-SIRS and k-SLIRS problems are NP-complete for every constant $k \geqslant 4$. The proof is based on a polynomial transformation to these problems from the following NP-complete problem.

Definition 7. The consecutive ones blocks problem for symmetric matrices (k-C1BS for short):
Instance: An $n \times n$ symmetric binary matrix M, and an integer $k > 0$.
Question: Is there a permutation of the rows of M such that for each column j the number of blocks of consecutive 1's (*i.e.*, the number of entries such that $M[i, j] = 1$ and either $M[i + 1, j] = 0$ or $i = n$) is at most k?

In [16], we proved that k-C1BS is NP-complete for every fixed $k \geqslant 2$.

In the next section, we show a construction of graphs from symmetric matrices such that the matrices can be row permuted to give each column not more than k blocks of consecutive 1's if and only if the constructed graph supports an all-shortest-path $(k+1)$-IRS ($(k+2)$-SIRS). In the final section, we will point out that the transformation can start from a similar NP-complete problem given in [5] to prove the NP-completeness of all-shortest-path k-IRS for general integer k, but not a constant k.

The NP-completeness for global compactness is proved in Section 4 by polynomial transformations from the following well-known NP-complete problem.

Definition 8. The Hamiltonian path problem (HP):
Instance: Graph $G = (V, E)$.
Question: Does G contain a Hamiltonian path?

3. NP-completeness for edge compactness

Starting with any instance of k-C1BS, $\langle M_{n\times n}, k\rangle$, where M is a symmetric binary matrix, we construct a graph $G = (V, E)$ such that there is a row permutation on M leading to each column having not more than k consecutive 1's blocks if and only if G supports an all-shortest-path $(k + 1)$-IRS. The construction is simple. For each row i of M, we create a set $R_i = \{r_{i,1}, r_{i,2}, \ldots, r_{i,k+4}\}$ of $k + 4$ vertices in G, which we call row vertices; for each column j of M, we create a set $C_j = \{c_{j,1}, c_{j,2}, \ldots, c_{j,2n(k+4)+1}\}$ of $2n(k + 4) + 1$ vertices in G, called column vertices; these two types of vertices induce a bipartite subgraph of G, and $R_i - C_j$ edges exist if and only if $M[i, j] = 1$. Finally, we add a new vertex a to G to link all the other vertices so that the diameter of G is at most 2. Formally, $G = (V, E)$ is obtained as follows (refer to Fig. 1, where for simplicity, a line between R_i and C_j represents an edge set, $\{(r_{i,l}, c_{j,h}) \mid r_{i,l} \in R_i,\ c_{j,h} \in C_j\}$).

$$
\begin{aligned}
V &= R + C + \{a\}, \quad \text{where} \\
&\quad R = \bigcup_{1\leqslant i\leqslant n} R_i \ \text{ and } \ R_i = \{r_{i,l} \mid 1 \leqslant l \leqslant k+4\}; \ \text{ and} \\
&\quad C = \bigcup_{1\leqslant j\leqslant n} C_j \ \text{ and } \ C_j = \{c_{j,h} \mid 1 \leqslant h \leqslant 2n(k+4)+1\}; \ \text{ and} \\
E &= \{(r_{i,l}, c_{j,h}) \mid r_{i,l} \in R_i,\ c_{j,h} \in C_j,\ M[i, j] = 1\} \\
&\quad \cup \{(a, r_{i,l}) \mid r_{i,l} \in R\} \cup \{(a, c_{j,h}) \mid c_{j,h} \in C\}.
\end{aligned}
$$

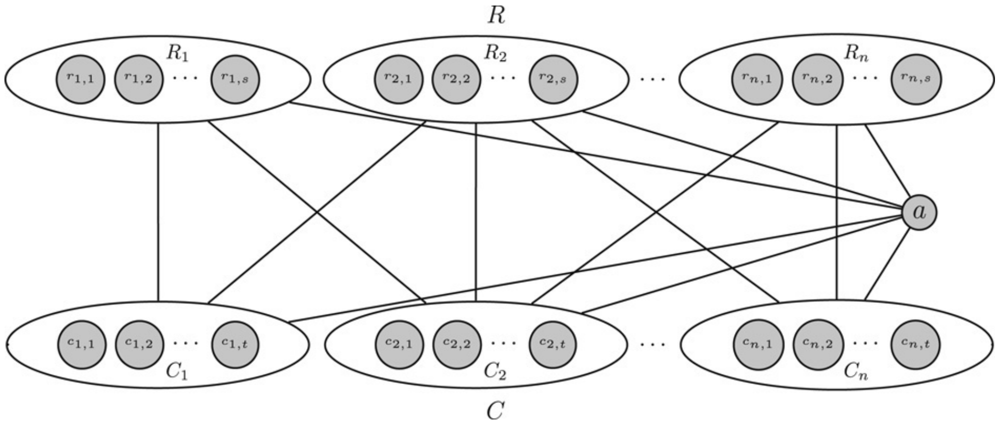

Fig. 1. The transformation graph ($s = k + 4$ and $t = 2n(k + 4) + 1$).

Note that any two vertices $r_{i,l}$ and $r_{i,l'}$ of R_i have identical neighbour sets $Adj(r_{i,l}) = Adj(r_{i,l'})$, and so do any two vertices $c_{j,h}$ and $c_{j,h'}$ of C_j. By $\Gamma(R_i)$ we denote the set of those column vertices that are neighbours of $r_{i,l} \in R_i$. Clearly, $\Gamma(R_i) = Adj(r_{i,l}) - \{a\} = \{c_{j,h} \mid (r_{i,l}, c_{j,h}) \in E\} = \bigcup_{M[i,j]=1} C_j$. Similarly, we use $\Gamma(C_j)$ to refer to the set of those row vertices that are linked with $c_{j,h} \in C_j$. Then, $\Gamma(C_j) = \bigcup_{M[i,j]=1} R_i$.

As the diameter of G is 2, for each arc $e = \langle u, v \rangle$, $S(e)$, the optimally reachable vertices from u via e, is $Adj(v) - Adj(u) + \{v\} - \{u\}$. We summarize in the next proposition the $S(e)$ for various types of arcs e in G.

Proposition 1. *In the transformation graph $G = (V, E)$,*

$$\begin{aligned} S(a,\ r_{i,l}) &= \{r_{i,l}\}, \\ S(a,\ c_{j,h}) &= \{c_{j,h}\}, \\ S(r_{i,l},\ a) &= \{a\} + R - \{r_{i,l}\} + C - \Gamma(R_i), \\ S(r_{i,l}, c_{j,h}) &= \{c_{j,h}\} + \Gamma(C_j) - \{r_{i,l}\}, \\ S(c_{j,h},\ a) &= \{a\} + C - \{c_{j,h}\} + R - \Gamma(C_j), \\ S(c_{j,h}, r_{i,l}) &= \{r_{i,l}\} + \Gamma(R_i) - \{c_{j,h}\}; \end{aligned}$$

for $1 \leqslant i, j \leqslant n$, $1 \leqslant l \leqslant k + 4$, and $1 \leqslant h \leqslant 2n(k + 4) + 1$.

In the above, as well as in the following, we use "+" and "−" for the union and the difference operation on sets respectively.

Lemma 1. *There exists a permutation of the rows of the symmetric matrix $M_{n\times n}$ that would result in a matrix having at most k blocks of consecutive 1's per column if and only if the graph G obtained by the transformation from $\langle M_{n\times n}, k\rangle$ supports an all-shortest-path $(k + 1)$-IRS.*

Proof. Suppose first of all that there exists a permutation π of the rows of $M_{n\times n}$ that leads to at most k blocks per column. Without loss of generality, assume $\pi(i) = i$; then the $(k + 1)$-IRS can be constructed as follows. The vertex labeling L is such that $L(r_{i,l}) = (i - 1)(k + 4) + l$ for each row vertex $r_{i,l} \in R$; $L(a) = n(k + 4) + 1$; and $L(c_{j,h}) = n(k+4)+1+(j-1)(2n(k+4)+1)+h$ for each column vertex $c_{j,h} \in C$. That is, the vertices are ordered in such a way that each $R_i \subset R$ forms one interval, each $C_j \subset C$ forms one interval, and so do R and C themselves; vertex a is in the middle and adjacent to the last vertex $r_{n,k+4}$ of R and the first vertex $c_{1,1}$ of C, as depicted below.

$$\underbrace{\overbrace{r_{1,1}\ldots r_{1,s}}^{R_1}\overbrace{r_{2,1}\ldots, r_{2,s}}^{R_2}\ldots\overbrace{r_{n,1}\ldots r_{n,s}}^{R_n}}_{R}\ a\ \underbrace{\overbrace{c_{1,1}\ldots c_{1,t}}^{C_1}\overbrace{c_{2,1}\ldots c_{2,t}}^{C_2}\ldots\overbrace{c_{n,1}\ldots c_{n,t}}^{C_n}}_{C}.$$

Because each R_i under labeling L forms a single interval, for each column j,

$$\begin{aligned} N(L, \Gamma(C_j)) &= N\left(L, \bigcup_{M[i,j]=1} R_i\right) = N\left(\pi, \{i \mid M[i,j]=1\}\right) \\ &= \text{the number of consecutive 1's blocks in the } j\text{th column of } M \\ &\leqslant k. \end{aligned}$$

For a similar reason and because of the symmetry of M, for each row i, we have

$$\begin{aligned} N(L, \Gamma(R_i)) &= N\left(L, \bigcup_{M[i,j]=1} C_j\right) = N\left(\pi, \{j \mid M[i,j]=1\}\right) \\ &= \text{the number of consecutive 1's blocks in the } i\text{th row of } M \\ &\leqslant k. \end{aligned}$$

Concerning the arc labeling I for all-shortest-path IRS, for each arc $e = \langle u, v\rangle$, there are only two alternatives: setting $I(u, v)$ to $S(u, v)$ or to $S(u, v) + u$. For different arcs, we make the choice that would guarantee each arc receiving not more than $k + 1$ intervals, as follows.

Vertex a**:** Let $I(a, r_{i,l}) = S(a, r_{i,l}) = \{r_{i,l}\}$ and $I(a, c_{j,h}) = S(a, c_{j,h}) = \{c_{j,h}\}$. Obviously each of them receives one interval. At other vertices, we will use self-enclosing intervals.

Vertex $r_{i,l}$**:** For arc $\langle r_{i,l}, a\rangle$, let $I(r_{i,l}, a) = S(r_{i,l}, a) + \{r_{i,l}\} = \{a\} + R + C - \Gamma(R_i) = V - \Gamma(R_i)$; then $N(L, I(r_{i,l}, a)) \leqslant N(\Gamma(R_i)) + 1 \leqslant k + 1$. For arc $\langle r_{i,l}, c_{j,h}\rangle$, let $I(r_{i,l}, c_{j,h}) = S(r_{i,l}, c_{j,h}) + \{r_{i,l}\} = \{c_{j,h}\} + \Gamma(C_j)$; then $N(L, I(r_{i,l}, c_{j,h})) \leqslant 1 + N(\Gamma(C_j)) \leqslant k + 1$.

Vertex $c_{j,h}$**:** Similarly, let $I(c_{j,h}, a) = S(c_{j,h}, a) + \{c_{j,h}\} = V - \Gamma(C_j)$ and $I(c_{j,h}, r_{i,l}) = S(c_{j,h}, r_{i,l}) + \{c_{j,h}\} = \{r_{i,l}\} + \Gamma(R_i)$; then each of these two types of arcs receives at most $k + 1$ intervals.

The *only if* part of the lemma is proved.

Suppose conversely that G admits an all-shortest-path $(k + 1)$-*IRS*, $\langle L, I\rangle$. Consider a permutation π on $\{1, 2, \ldots, n\}$ induced from L such that

$$\pi(i) < \pi(j) \iff \min\{L(v) \mid v \in R_i\} < \min\{L(v) \mid v \in R_j\}.$$

We need only to justify that, for every $1 \leqslant j \leqslant n$, $N(\pi, \{i \mid M[i, j] = 1\})$, the number of blocks in the jth column of M, after the rows have been permuted using π, is not more than k.

For $1 \leqslant i \leqslant n$, let $\min R_i$ denote the minimal vertex in R_i, *i.e.*, $L(\min R_i) = \min\{L(r_{i,l}) \mid r_{i,l} \in R_i\}$. If $M[i_1, j] = 1$ and $M[i_2, j] = 1$, and the two vertices $\min R_{i_1}$ and $\min R_{i_2}$ belong to the same interval of $\bigcup_{M[i,j]=1} R_i$ under L, then in the matrix M, which is permuted according to π, the two rows i_1 and i_2 must belong to the same block of consecutive 1's. Thus we have $N(\pi, \{i \mid M[i, j] = 1\}) \leqslant N(L, \bigcup_{M[i,j]=1} R_i) = N(L, \Gamma(C_j))$. So we need only to prove that $N(L, \Gamma(C_j)) \leqslant k$ for every $1 \leqslant j \leqslant n$.

We first show that for any column j, $N(L, \Gamma(C_j)) \leqslant k + 3$. Otherwise, there would be some j such that $N(L, \Gamma(C_j)) > k + 3$. Consider the interval number $N(L, I(r_{i,l}, c_{j,h}))$ on an arc $\langle r_{i,l}, c_{j,h}\rangle$ at a vertex $r_{i,l} \in \Gamma(C_j)$. $I(r_{i,l}, c_{j,h})$ is either $S(r_{i,l}, c_{j,h})$ or $S(r_{i,l}, c_{j,h}) + \{r_{i,l}\}$. If it is $S(r_{i,l}, c_{j,h})$, then

$$N(L, I(r_{i,l}, c_{j,h})) = N(L, \{c_{j,h}\} + \Gamma(C_j) - \{r_{i,l}\}) \geqslant N(L, \Gamma(C_j)) - 2 > k + 1.$$

If it is $S(r_{i,l}, c_{j,h}) + \{r_{i,l}\}$, then

$$N(L, I(r_{i,l}, c_{j,h})) = N(L, \{c_{j,h}\} + \Gamma(C_j)) \geqslant N(L, \Gamma(C_j)) - 1 > k + 2.$$

Both cases imply that under the labeling L the edge $(r_{i,l}, c_{j,h})$ must receive more than $k + 1$ intervals, contradicting that $\langle L, I\rangle$ is a $(k + 1)$-IRS for G.

We now prove that for any column j, $N(L, \Gamma(C_j)) \leqslant k$. Supposing that it is not the case, then there must be a j such that $N(L, \Gamma(C_j)) \geqslant k + 1$. In this case, we can show that there exist an $r_{i,l} \in \Gamma(C_j)$ and a $c_{j,h} \in C_j$ such that the arc $\langle r_{i,l}, c_{j,h}\rangle$ at vertex $r_{i,l}$ receives at least $k + 2$ intervals, contradicting that L is a vertex labeling of the $(k + 1)$-*IRS*.

Since $|C_j| = 2n(k+4)+1 = 2|R|+1 \geqslant 2|\Gamma(C_j)|+1$, and each $r_{i,l} \in \Gamma(C_j)$ is adjacent to not more than two vertices of C_j (with respect to the order defined by the labeling L), there must be a vertex $c_{j,h} \in C_j$ such that $L(c_{j,h})$ is adjacent to no labels of the vertices in $\Gamma(C_j)$. In the set $S(r_{i,l}, c_{j,h}) = \{c_{j,h}\} + \Gamma(C_j) - \{r_{i,l}\}$, under the labeling L, $c_{j,h}$ itself has to form a single interval. If $I(r_{i,l}, c_{j,h})$ is $S(r_{i,l}, c_{j,h}) + \{r_{i,l}\}$, then the interval number of arc $\langle r_{i,l}, c_{j,h}\rangle$ is

$$N(L, I(r_{i,l}, c_{j,h})) = N(L, \{c_{j,h}\} + \Gamma(C_j)) = 1 + N(L, \Gamma(C_j)) \geqslant k+2,$$

which is a contradiction. If $I(r_{i,l}, c_{j,h})$ is $S(r_{i,l}, c_{j,h})$, by further selecting an appropriate $r_{i,l}$ from $\Gamma(C_j)$ we can arrive at another contradiction. Because there are at least $k+4$ vertices in $\Gamma(C_j)$ and all of the vertices of $\Gamma(C_j)$ are distributed among the $N(L, \Gamma(C_j)) \leqslant k+3$ intervals, there must be an interval $[L(r_{i,l}), L(r_{i',l'})]$ of $\Gamma(C_j)$ having at least two vertices. Selecting the boundary vertex $r_{i,l}$ in this interval, we have $N(L, \Gamma(C_j) - \{r_{i,l}\}) = N(L, \Gamma(C_j))$. Hence the number of the intervals assigned on edge $(r_{i,l}, c_{j,h})$ is

$$\begin{aligned} N(L, I(r_{i,l}, c_{j,h})) &= N(L, \{c_{j,h}\} + \Gamma(C_j) - \{r_{i,l}\}) \\ &= 1 + N(L, \Gamma(C_j) - \{r_{i,l}\}) = 1 + N(L, \Gamma(C_j)) \\ &\geqslant k+2, \end{aligned}$$

which is a contradiction. The *if* part of the lemma is proved. □

It is easy to see that the above transformation can be performed in polynomial time. By the NP-completeness of k-C1BS for every fixed $k \geqslant 2$, the next theorem follows.

Theorem 1. *Given a network G, the problem of deciding if there exists an all-shortest-path k-IRS for G is NP-complete for every fixed $k \geqslant 3$.*

Note that, in the *only if* part of the proof of Lemma 1, the $(k+1)$-IRS for G derived from the permutation of the rows of M is linear. Thus, the following holds.

Theorem 2. *Given a network G, the problem of deciding if there exists an all-shortest-path k-LIRS for G is NP-complete for every fixed $k \geqslant 3$.*

In the transformation, if we make each R_i contain $2k+7$ vertices and each C_j contain $2n(2k+7)+1$ vertices, we would be able to find one of the intervals of $\Gamma(C_j)$ under L containing at least three vertices. Selecting an intermediate vertex from such an interval, a vertex $r_{i,l} \in \Gamma(C_j)$ can be found satisfying $N(L, \Gamma(C_j) - \{r_{i,l}\}) = N(\Gamma(C_j)) + 1$. Note that in all-shortest-path SIRS, the arc labeling L is identical to S. By a similar argument one can show that there exists an all-shortest-path strict $(k+2)$-IRS for G if and only if there exists a permutation for M leading to at most k consecutive 1's blocks per column.

Theorem 3. *Given a network G, the problem of deciding if there exists an all-shortest-path k-SIRS (k-SLIRS) for G is NP-complete for every fixed $k \geqslant 4$.*

4. NP-completeness for global compactness

Flammini et al. in [4,5] proved that the all-shortest-path global k-IRS and SIRS problems are NP-complete in the weighted graph. In this section we strengthen their results to cover unweighted graphs and extend them to global k-LIRS and SLIRS. We obtain these results by a transformation from the Hamiltonian path problem. The transformation, presented in Section 4.1, is simple, but the correctness proof, presented in Sections 4.2 and 4.3, is not.

4.1. Transformation from Hamiltonian path

Let $G = (V, E)$, where $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{e_1, e_2, \ldots, e_m\}$, be an instance of the Hamiltonian path problem. We construct a graph $\mathbb{G} = (\mathbb{V}, \mathbb{E})$, as follows (see Fig. 2): Every vertex v_k and every edge e_i of G become vertices of $\mathbb{G}$, in which vertex e_i is linked to the two vertices v_k, v_h corresponding to the edge $e_i = (v_k, v_h)$ in G; in order to force the optimal solution to be such that the labels of the vertices in E would not be adjacent to each other nor to the labels of those in V, $m+1$ cliques $\mathbb{B}_0, \mathbb{B}_1, \ldots, \mathbb{B}_m$ are added to $\mathbb{G}$, and each e_i is linked to a specific vertex

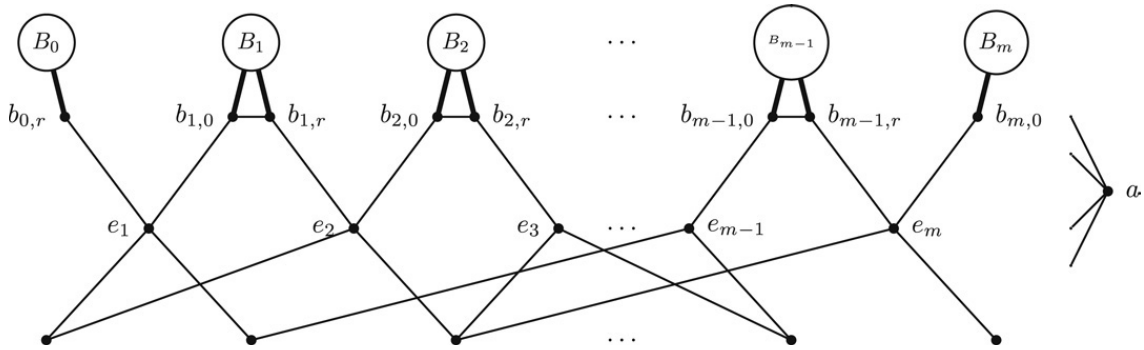

Fig. 2. The transformation graph $\mathbb{G} = (\mathbb{V}, \mathbb{E})$ (each B_i is a clique of size $r - 1$, and the thick line between $b_{i,j}$ and B_i represents the edges between $b_{i,j}$ and all vertices of B_i).

in $\mathbb{B}_{i-1}$ and a specific vertex in $\mathbb{B}_i$; finally, a special vertex a is added to link with all the other vertices so that the shortest path between any pair of vertices is at most 2. Formally, let $r = 13m + n - 1$, we have

$$
\begin{aligned}
\mathbb{V} &= \{a\} + \mathbb{B} + E + V, \quad \text{where} \\
&\quad \mathbb{B} = \bigcup_{0 \leqslant i \leqslant m} \mathbb{B}_i \quad \text{with} \\
&\quad\quad \mathbb{B}_0 = B_0 + \{b_{0,r}\}; \quad \mathbb{B}_m = \{b_{m,0}\} + B_m; \\
&\quad\quad \mathbb{B}_i = \{b_{i,0}\} + B_i + \{b_{i,r}\} \ \text{ for } \ 0 < i < m; \\
&\quad\quad B_i = \{b_{i,j} \mid 0 < j < r\} \ \text{ for } \ 0 \leqslant i \leqslant m; \\
\mathbb{E} &= E_A + E_{BB} + E_{BE} + E_{EV}, \quad \text{where} \\
&\quad E_A \ = \{(a, u) \mid u \in \mathbb{V}, \ u \neq a\}; \\
&\quad E_{BB} = \{(b_{i,j}, b_{i,k}) \mid 0 \leqslant i \leqslant m, \ j \neq k\}; \\
&\quad E_{BE} = \{(b_{i,r}, e_{i+1}) \mid 0 \leqslant i \leqslant m-1\} \cup \{(e_i, b_{i,0}) \mid 1 \leqslant i \leqslant m\}; \\
&\quad E_{EV} = \{(e_i, v_k) \mid e_i = (v_k, u)\}.
\end{aligned}
$$

For various types of arcs, e, in the constructed $\mathbb{G}$, $S(e)$ is summarized in Table 2, in which the arcs are classified into three classes; each class is further divided into several types. Note that the notation e_i is multi-purpose—we use it to refer to an edge $e_i = (v_k, v_h)$ in graph G, a vertex in graph $\mathbb{G}$, and sometimes the set $\{v_k, v_h\}$, depending on the context it appears. For example, the "e_i" in "$v_k \in e_i$" refers to a set of two vertices. For $v_k \in V$, we use $\Gamma(v_k)$ to denote the set of those vertices $e_i \in E$ which are linked to v_k in $\mathbb{G}$; take v_3 in Fig. 2 for instance, $\Gamma(v_3) = \{e_2, e_3, e_m\}$; clearly, $\Gamma(v_k) = \{e_i \mid v_k \in e_i\}$, and the degree of v_k in G, $d_G(v_k) = |\Gamma(v_k)| = d_{\mathbb{G}}(v_k) - 1$ (because in $\mathbb{G}$, v_k is linked to a).

4.2. The NP-completeness

It is easy to see that for a graph $G = (V, E)$ with $|V| = n$ and $|E| = m$, the corresponding graph $\mathbb{G} = (\mathbb{V}, \mathbb{E})$ can be constructed in polynomial time of m, n. We need to prove that G has a Hamiltonian path if and only if $\mathbb{G}$ supports an all-shortest-path IRS (SIRS, LIRS, SLIRS, respectively) with global compactness K, where K is determined by G and the variant of IRS.

We refer to a vertex labeling L for $\mathbb{G}$ as *normal* if it labels the members of $\mathbb{B} + E$ consecutively in the following order (or the reversed).

$$B_0 b_{0,r} e_1 b_{1,0} B_1 b_{1,r} e_2 b_{2,0} B_2 b_{2,r} e_3 b_{3,0} B_3 b_{3,r} \ldots e_{m-1} b_{m-1,0} B_{m-1} b_{m-1,r} e_m b_{m,0} B_m. \tag{1}$$

In the above, the vertices of each B_i and each $\mathbb{B}_i$ are labeled consecutively by the normal labeling.

We first claim that the optimal global compactness of $\mathbb{G}$ for all-shortest-path IRS (SIRS, LIRS, SLIRS) can always be reached with a normal vertex labeling, as expressed in the following lemma.

Lemma 2. *For any vertex labeling L on $\mathbb{V}$, there is a normal vertex labeling L' on $\mathbb{V}$ such that $N(L') \leqslant N(L)$.*

Table 2
$S(e)$ for arcs e in $\mathbb{G} = (\mathbb{V}, \mathbb{E})$ ($\{b_{0,0}\} = \{b_{m,r}\} = \phi$)

Class		Arc e	Scope	$S(e)$	Number of arcs	
I		(a, u)	$u \in \mathbb{V}$, $u \neq a$	$\{u\}$	$(r+1)(m+1)+m+n-2$	
I		(b_{ij}, b_{ik})	$b_{i,j} \in \mathbb{B}_i$, $b_{i,k} \in B_i$, $b_{i,j} \neq b_{i,k}$	$\{b_{i,k}\}$	$(r^2-r)(m-1)+2(r-1)^2$	
II	1	$(b_{i,0}, b_{i,r})$	$0 < i < m$	$\{b_{i,r}, e_{i+1}\}$	$m-1$	$r-1$
		$(b_{i,r}, b_{i,0})$	$0 < i < m$	$\{e_i, b_{i,0}\}$	$m-1$	
		$(e_i, b_{i,0})$	$0 < i \leqslant m$	$\{b_{i,0}\} \cup B_i \cup \{b_{i,r}\}$	m	
		(e_i, b_{i-1}, r)	$0 < i \leqslant m$	$[\{b_{i-1,0}\} \cup B_{i-1} \cup \{b_{i-1,r}\}]$	m	
	2	$(b_{i,0}, a)$	$0 < i \leqslant m$	$\mathbb{V} - [\{e_i, b_{i,0}\} \cup B_i \cup \{b_{i,r}\}]$	m	
		$(b_{i,r}, a)$	$0 \leqslant i < m$	$\mathbb{V} - [\{b_{i,0}\} \cup B_i \cup \{b_{i,r}, e_{i+1}\}]$	m	
	3	$(b_{i,0}, e_i)$	$0 < i \leqslant m$	$\{b_{i-1,r}, e_i\} \cup e_i$	m	
		$(b_{i,r}, e_{i+1})$	$0 \leqslant i < m$	$\{e_{i+1}, b_{i+1,0}\} \cup e_{i+1}$	m	
		(e_i, a)	$0 < i \leqslant m$	$\mathbb{V} - [\{b_{i-1,r}, e_i, b_{i,0}\} \cup e_i]$	m	
	4	(e_i, v_k)	$v_k \in e_i$	$\{v_k\} \cup \Gamma(v_k) - \{e_i\}$	$2m$	
		(v_k, a)	$1 \leqslant k \leqslant n$	$\mathbb{V} - [\{v_k\} \cup \Gamma(v_k)]$	n	
		(v_k, e_i)	$v_k \in e_i$	$\{b_{i-1,r}, e_i, b_{i,0}\} \cup e_i - \{v_k\}$	$2m$	
III	1	$(b_{i,j}, b_{i,0})$	$0 < i \leqslant m$, $0 < j < r$	$\{e_i, b_{i,0}\}$	$(r-1)m$	
		$(b_{i,j}, b_{i,r})$	$0 \leqslant i < m$, $0 < j < r$	$\{b_{i,r}, e_{i+1}\}$	$(r-1)m$	
	2	$(b_{i,j}, a)$	$0 \leqslant i \leqslant m$, $0 < j < r$	$\mathbb{V} - [\{b_{i,0}\} \cup B_i \cup \{b_{i,r}]$	$(r-1)(m+1)$	

The proof of this lemma is deferred to the next subsection. With this lemma, we can consider only normal vertex labeling L of $\mathbb{G}$. In the rest of this subsection, we always assume L to be normal, and let l be the number of vertices $e_i = (v_k, v_h) \in E \subset \mathbb{V}$ such that v_k and v_h receive adjacent labels under L. We have the following facts.

Fact 1: $l \leqslant n-1$, and the equality holds only if the graph $G = (V, E)$ contains a Hamiltonian path.
Fact 2: Any arc e of Class I has a single vertex in its $S(e)$; thus independently with any vertex labeling, it always receives a single interval. Let $s = (r+1)(m+1)+m+n-2+(r^2-r)(m-1)+2(r-1)^2$, the number of the arcs in Class I, then the arcs of Class I contribute a total of s intervals.
Fact 3: Under L, each arc of Class II.1 or III.1 receives a single interval. Thus the arcs of Class II.1 and III.1 totally yield $t = 4m - 2 + 2(r-1)m$ intervals.
Fact 4: Under L, the labels of vertices of E are pairwise separated, and none of them is adjacent to the labels of $v \in V$; also, labels of vertices of V are not adjacent to $L(b_{i,0})$ or $L(b_{i,r})$.

To arrive at the NP-completeness, we need the following lemma.

Lemma 3. *There is an integer K such that $G = (V, E)$ contains a Hamiltonian path if and only if the transformation graph $\mathbb{G} = (\mathbb{V}, \mathbb{E})$ constructed from G supports an all-shortest-path global K-IRS (SIRS, LIRS, SLIRS, respectively).*

Proof. By Lemma 2, we can restrict our attention to normal labelings only. The idea is to show that under any normal labeling L, the total number of intervals of $\mathbb{G}$ is $f(m, n) - g(l)$, where f and g are strictly increasing polynomial functions. Thus, by Fact 1, we can conclude that G has a Hamiltonian path if and only if $\mathbb{G}$ supports an all-shortest-path IRS of global compactness $K = f(m, n) - g(n-1)$. We have counted the interval numbers contributed by Classes I, II.1 and III.1. For other arcs, we need to consider the differences among IRS variants.

Let us consider SIRS first. Note that SIRS allows circular intervals, which means $N(L, \mathbb{V} - U) = N(L, U)$. Under L, we have in

Class II.2, $2m$ intervals, one per arc.
Class II.3, $9m - 3l$ intervals, because by Fact 4, each arc receives $1 + N(L, e_i)$ intervals, totally $3 \times t \sum_{0<i\leqslant m}(1 + N(L, e_i)) = 3(m + 2(m-l) + l) = 9m - 3l$.

Class II.4, $\sum_{1\leqslant k\leqslant n} d_G^2(v_k) + 4m + n$ intervals, because
$N(L, S(e_i, v_k)) = N(L, \{v_k\} \cup \Gamma(v_k) - \{e_i\}) = d_G(v_k)$, and the total number of intervals contributed by arcs from E to V is $\sum_{e_i\in\Gamma(v_k)} d_G(v_k) = \sum_{1\leqslant k\leqslant n} d_G^2(v_k)$;
$N(L, S(v_k, a)) = N(L, \mathbb{V} - [\{v_k\} \cup \Gamma(v_k)]) = |\{v_k\} \cup \Gamma(v_k)| = d_G(v_k) + 1$, and there are $\sum_{1\leqslant k\leqslant n}(d_G(v_k) + 1) = 2m + n$ intervals on arcs from V to vertex a;
$N(L, S(v_k, e_i)) = N(L, \{b_{i-1,r}, e_i, b_{i,0}\} \cup e_i - \{v_k\}) = 2$, and so $2m$ arcs from V to E contribute $4m$ intervals.
Class III.2 , $(r - 1)(m + 1)$ intervals, one per arc.

Thus, the total number of intervals for $\mathbb{G}$ under normal labeling L is

$$\begin{aligned} & s + t + 2m + 8m - 3l + \sum_{1\leqslant k\leqslant n} d_G^2(v_k) + 4m + n + (r - 1)(m + n) \\ & \quad = \sum_{1\leqslant k\leqslant n} d_G^2(v_k) + s + t + (r - 1)(m + n) + 14m + n - 3l. \end{aligned}$$

Let

$$K = \sum_{1\leqslant k\leqslant n} d_G^2(v_k) + s + t + (r - 1)(m + n) + 14m + n - 3(n - 1),$$

then $\mathbb{G} = (\mathbb{V}, \mathbb{E})$ admits an all-shortest-path SIRS of global compactness K if and only if $l \geqslant n - 1$ and $G = (V, E)$ contain a Hamiltonian path. The lemma for SIRS is proved.

Now consider IRS, for which we can use self-enclosing intervals to reduce the number of intervals needed. Nevertheless, under normal labeling, using self-enclosing intervals can only help those arcs $\langle v_k, a\rangle$ of Class II.4—by assigning $I(v_k, a) = S(v_k, a) + \{v_k\} = \mathbb{V} - \Gamma(v_k)$, the number of intervals on each arc $\langle v_k, a\rangle$ is reduced by one. Compared with SIRS, IRS can save totally and exactly n intervals. Setting K to be n less than the K for SIRS, we can claim the same result for IRS. So the lemma for IRS is proved.

For LIRS or SLIRS, if U involves no complement operation with respect to $\mathbb{V}$, then $N(L, U)$ has the same value as for IRS or SIRS. $N(L, \mathbb{V} - U)$ can be $N(L, U) - 1$, $N(L, U)$, or $N(L, U) + 1$, corresponding to 2, 1, or 0 vertices respectively in U with label 1 or $|\mathbb{V}|$. That is, calling $v \in \mathbb{V}$ a terminal vertex, if $L(v) = 1$ or $|\mathbb{V}|$, then for LIRS or SLIRS we have

$$N(L, \mathbb{V} - U) = N(L, U) + 1 - \text{ the number of terminals in } U. \tag{2}$$

We need to analyse which vertices of $\mathbb{V}$ may possibly be labeled as terminals by an optimal normal labeling L.

Since L is normal, by (1), the two terminal vertices can only come from $B_0 \cup B_m \cup V \cup \{a\}$, and at most one of them comes from $B_0 \cup B_m$. Hence we can assume that the two terminals come from $B_0 \cup V \cup \{a\}$. According to (2), for a vertex v that is qualified to be a terminal, there must be as many subsets $S(e) = \mathbb{V} - U$ as possible for v in U. From Table 2, all the subset $S(e)$'s involving a complement operation take the form $\mathbb{V} - U$ with $a \notin U$; thus vertex a can be excluded. So, we can assume that the two terminals come from $B_0 \cup V$ and at most one of them comes from B_0. Further analysis needs a closer look at those $S(e)$'s that are computed in the form $\mathbb{V} - U$ with $U \cap (B_0 \cup V) \neq \phi$. They are:

$$\begin{aligned} S(e_i, a) &= \mathbb{V} - (\{b_{i-1,r}, e_i, b_{i,0}\} \cup e_i) \quad \text{for } 0 < i \leqslant m, \\ S(v_k, a) &= \mathbb{V} - (\{v_k\} \cup \Gamma(v_k)) \quad \text{for } 1 \leqslant k \leqslant n, \\ S(b_{0,r}, a) &= \mathbb{V} - (B_0 \cup \{b_{0,r}, e_1\}), \quad \text{and} \\ S(b_{0,j}, a) &= \mathbb{V} - (B_0 \cup \{b_{0,r}\}) \quad \text{for } 0 < j < r. \end{aligned}$$

There are $r = 13m + n - 1$ subset $S(e)$'s taking the form $\mathbb{V} - U$ such that U intersects with B_0 but not V (the last two of the above). The other $m + n$ subset $S(e)$'s take the form $\mathbb{V} - U$ such that U intersects with V but not B_0. Of course, to save more intervals, one (and the only one) terminal should be from B_0. Without loss of generality, we can assume it to be $b_{0,1}$. For any $v_k \in V$, there are $d_G(v_k)$ subsets $S(e_i, a) = \mathbb{V} - U$ and one subset $S(v_k, a) = \mathbb{V} - U$ such that $v_k \in U$, *i.e.*, v_k appears in $d_G(v_k) + 1$ subset U's of the required form. So the other terminal should be the one with maximum degree in G. Let that be v_h; then $d_G(v_h) = \Delta = \max\{d_G(v_k) \mid v_k \in V\}$.

There are $3m+n+(r-1)(m+1)$ arcs e in $\mathbb{G}$ whose $S(e)$ involves the complement operation. From the above analysis, we are sure that among these arcs, there are exactly $r+\Delta+1$ arcs for SLIRS ($r+\Delta$ arcs for LIRS, resp.) receiving the same number of intervals as they do in SIRS (IRS, resp.); each of the other $3m+n+(r-1)(m+1)-(r+\Delta+1)$ arcs ($3m+n+(r-1)(m+1)-(r+\Delta)$ arcs, resp.) receives one more interval than it does in SIRS (IRS, resp.). Thus, let K be $3m+n+(r-1)(m+1)-(r+\Delta+1)$ larger than the K we have determined for SIRS (resp. $3m+n+(r-1)(m+1)-(r+\Delta)$ larger than the K we have determined for SIRS), we conclude that the lemma statement is true for SLIRS (LIRS). $\square$

By Lemma 3, we have the following theorem.

Theorem 4. *Given a network G and an integer K, the problem of deciding if there exists an all-shortest-path IRS (SIRS, LIRS, SLIRS, respectively) for G with global compactness of at most K is NP-complete.*

4.3. *The proof of Lemma 2*

The proof consists of two steps. The first step is to argue that any vertex labeling L on $\mathbb{V}$ can be converted into a labeling L' which is not worse than L in terms of overall number of intervals used and which labels each B_i with a single interval.

Arcs of Class I can be excluded from calculating the overall interval number because each of them, independent of the vertex labeling, always receives a single interval. For any arc e belonging to the other two classes, either $B_i \subseteq S(e)$ or $B_i \subseteq \mathbb{V} - S(e)$. So, given any vertex labeling L, the labeling L' is obtained from L by compacting all the labels of the vertices in B_i in a unique interval and leaving unchanged the relative order of the labels of all the other vertices. This yields a scheme requiring an overall number of intervals at most equaling to that of the scheme using L (*i.e.*, $N(L') \leqslant N(L)$). Hence in the following we restrict our attention only to such a vertex labeling, which we call *bounded* labeling.

The next step is to show that any bounded vertex labeling L can be converted into a normal vertex labeling L' which is no worse than L. This would complete the proof of Lemma 2. To do this, we need the following definitions.

Definition 9. Suppose L is a vertex labeling on $\mathbb{V}$. We call a subset $U \subseteq \mathbb{B} \cup E$ a normal set with respect to L (or say L is locally normal with respect to U) if L labels the vertices of U consecutively of the order of a normal labeling. We call U a *maximal* normal set if U is normal, but for any $U' \supset U$, U' is not. If U is normal, we use $MaxNorm(L, U)$ to denote the maximal normal superset set of U, i.e., $MaxNorm(L, U) \supseteq U$ and $MaxNorm(L, U)$ is maximal normal. Since a normal set U forms a single interval under L, we also call U a normal interval.

The idea is to show that, for any bounded labeling L on $\mathbb{V}$, if $MaxNorm(L, B_0) \neq \mathbb{B} \cup E$, then we can repeatedly augment $MaxNorm(L, B_0)$ by applying a change to L. The change is to "move" $MaxNorm(L, B_0)$ to merge with some other maximal normal set without bringing in extra intervals, to eventually obtain a labeling L' which is not worse than L, satisfies $MaxNorm(L, B_0) = \mathbb{B} \cup E$, and therefore is normal. The movement operation on L is defined below.

Definition 10. Suppose $U \subseteq \mathbb{V}$ and L assign to U a single interval with $u \in U$ as one of the two endpoints of the interval, $W \subseteq \mathbb{V}$ and L assign to W a single interval with an endpoint $w \in W$, and $U \cap W = \phi$. Then $Move(U, W, u, w)$ is the operation on L that moves the interval U to be next to the interval W, making u adjacent to w but leaving the relative order of the vertices in U unchanged (or reversed in order to make u and v adjacent). The new vertex labeling L' as a result is denoted by $Move(L, U, W, u, w)$.

Fig. 3 shows an example of the $Move(U, W, u_1, w_1)$ operation which needs to reverse U first. Note that if $Move(U, W, u_1, w_l)$ or $Move(U, W, u_h, w_1)$ is performed, the operation will not reverse U. The following lemma says that such a change causes at most one more interval.

Lemma 4. *Let L be a vertex labeling on $\mathbb{V}$, under which $U \subseteq \mathbb{V}$ forms a single interval with an endpoint $u \in U$, $W \subseteq \mathbb{V}$ forms a single interval with an endpoint $w \in W$, $U \cap W = \phi$, and $L' = Move(L, U, W, u, w)$; then $N(L', X) \leqslant N(L, X) + 1$ for any subset $X \subseteq \mathbb{V}$.*

Proof. (refer to Fig. 3) Suppose L labels V of the order of

$$\ldots, y_1, u_1, u_2, \ldots, u_h, y_2, \ldots, z_1, w_1, w_2, \ldots, w_l, z_2, \ldots$$

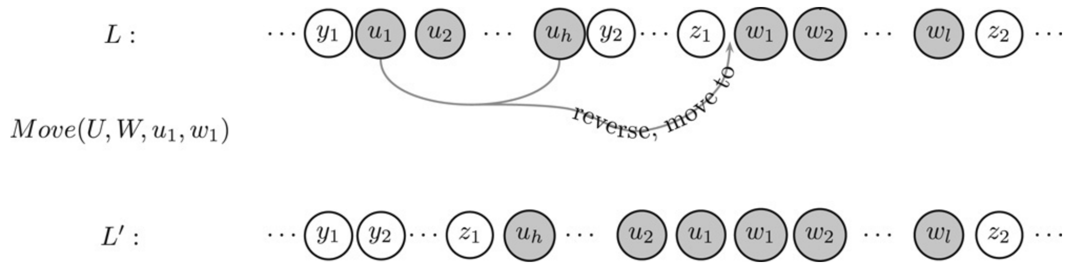

Fig. 3. $Move(U, W, u_1, w_1)$ on L to get L'.

in which, $U = \{u_1, u_2, \ldots, u_h\}$ and $W = \{w_1, w_2, \ldots, w_l\}$. Without loss of generality, assume $u = u_1$ and $w = w_1$. Then, we consider the three sets: $\{y_1, u_1\}$, $\{u_h, y_2\}$, and $\{z_1, w_1\}$. Only in the case that at least two of the three sets are subsets of X can the movement $Move(L, U, W, u, w)$ break two intervals of X into four broken fragments and thus possibly make $N(L', X)$ at least larger than $N(L, X)$ by two. On the other hand, if this happens, then we have the following (referring to Fig. 3).

- If only two of the three sets are subsets of X, then two of the four broken fragments will be merged into one; for examples, if both $\{y_1, u_1\}$ and $\{u_h, y_2\}$ are subsets of X, then the broken fragment containing y_1 and the one containing y_2 would be merged into one interval of X under L'; if one of $\{y_1, u_1\}$ and $\{u_h, y_2\}$, say $\{y_1, u_1\}$, and $\{z_1, w_1\}$ are subsets of X, then the broken fragment containing u_1 and the one containing w_1 will be merged into one interval of X under L'.
- If the three sets are all subsets of X, then L' will merge the broken fragment containing y_1 with the one containing y_2, the one containing u_1 with the one containing w_1, and the one containing w_h with the one containing z_1.

In summary, if the movement breaks i intervals of X into $2i$ fragments, then it will also merge $i - 1$ pairs of the broken fragments. Thus, we have $N(L', X) \leqslant N(L, X) + 1$. $\square$

Now we can proceed to the second step of the proof. Let L be a bounded vertex labeling on $\mathbb{V}$, suppose $U = MaxNormal(L, B_0) \neq \mathbb{B} \cup E$, and assume $b_{0,1} \in B_0$ and $u \in U$ be the two endpoints of U under L. Then u can only be e_k, or $b_{k,0}$, or some $v \in B_k$, say $b_{k,1}$, or $b_{k,r}$.

1. $u = e_k$: Under L, e_k is adjacent to $b_{k-1,r}$ but not $b_{k,0}$. Let $W = MaxNormal(L, \{b_{k,0}\})$ with endpoints $b_{k,0}$; the following depicts one of the possible cases of $\mathbb{V}$ under L.

$$\ldots \underbrace{B_0 b_{0,r} e_1 b_{1,0} B_1 \ldots B_{k-1} b_{k-1,r} e_k}_{U = MaxNormal(L, B_0)}\ x \ldots\ B_s \ldots y\ \underbrace{b_{k,0} B_k b_{k,r} e_{k+1} \ldots w}_{W = MaxNormal(L, \{b_{k,0}\})}\ z \ldots B_t \ldots.$$

   We make e_k and $b_{k,0}$ adjacent to form a larger normal interval $U \cup W$ under labeling $L' = Move(L, U, W, e_k, b_{k,0})$. We have $N(L', \{e_k, b_{k,0}\}) = 1 = N(L, \{e_k, b_{k,0}\}) - 1$, a reduction by one interval on each arc $\langle b_{i,j}, b_{i,0}\rangle$ of Class III.1 with $i = k$ and $0 < j < m$. For any other arc e of Class III, we have $N(L', S(e)) \leqslant N(L, S(e))$ because this movement does not change the adjacency between $b_{i,r}$ and e_{i+1} or among $b_{i,0}$, B_i and $b_{i,r}$. Hence the overall number of intervals for arcs of Class III is reduced by $r - 1 = 13m + n - 2$ which equals the number of arcs in Class II. Therefore, $N(L') \leqslant N(L)$ by Lemma 4.
2. $u = b_{k,0}$: B_k is not adjacent to $b_{k,0}$ under L. Let $W = MaxNormal(L, B_k)$ with endpoint $b_{k,1} \in B_k$. By $L' = Move(L, U, W, b_{k,0}, b_{k,1})$ we make $b_{k,0}$ and $b_{k,1}$ adjacent and merge U and W into a larger normal interval $U \cup W$. We have $N(L', \{b_{k,0}\} \cup B_k \cup \{b_{k,r}\}) = N(L, \{b_{k,0}\} \cup B_k \cup \{b_{k,r}\}) - 1$, one interval less on each arc $\langle b_{k,j}, a\rangle$ (of Class III.2) for $0 < j < r$. For any other arc e of Class III, we have $N(L', S(e)) \leqslant N(L, S(e))$ because this movement does not change the adjacency between e_i and $b_{i,0}$ or between $b_{i,r}$ and e_{i+1}. Hence, as argued in the above case, $MaxNormal(L', B_0) = U \cup W$ and $N(L') \leqslant N(L)$.
3. $u = b_{k,1}$: B_k is not adjacent to $b_{k,r}$. Let $W = MaxNormal(L, \{b_{k,r}\})$ with endpoint $b_{k,r}$. Similar to the previous case, $Move(U, W, b_{k,1}, b_{k,r})$ makes B_k adjacent to $b_{k,r}$ and produces a larger normal interval $U \cup W$ under labeling $L' = Move(L, U, W, b_{k,0}, b_{k,1})$. In the same manner, this movement saves at least $r - 1$ intervals overall on all arcs of Class III. Thus L' is not worse than L and is closer to a normal labeling than L.
4. $u = b_{k,r}$: $b_{k,r}$ is not adjacent to e_{k+1}. Let $W = MaxNormal(L, \{e_{k+1}\})$. We can argue similarly as in the first case that $Move(U, W, b_{k,r}, e_{k+1})$ makes a larger normal interval $U \cup W$ under labeling $L' = Move(L, U, W, b_{k,r}, e_{k+1})$,

reducing the number of intervals by $r-1$ on arcs $\langle b_{k,j}, b_{k,r}\rangle$ and introducing no additional intervals on the other arcs of Class III.

Thus $MaxNormal(L', B_0) = U \cup W$ and $N(L') \leqslant N(L)$, and the proof of Lemma 2 is complete.

5. Discussions

We have proved that to recognize networks that admit all-shortest-path k-IRS (k-SIRS) for every $k \geqslant 3$ ($k \geqslant 4$) is NP-complete for unweighted graphs, and of course also for weighted graphs. Our transformation takes advantage of the symmetry of the matrix in the NP-complete problem k-C1BS. For general binary matrices, Booth and Lueker [2] gave a linear algorithm for $k = 1$; Goldberg et al. [10] proved NP-completeness for every fixed $k \geqslant 2$; Flammini et al. [5] showed the same for general k even if the matrices are restricted to each row having not more than k blocks of consecutive 1's (although that is not stated explicitly in their paper). If we apply our transformation to an arbitrary binary m by n matrix, a graph will be constructed such that the matrix has not more than k blocks in each column and not more than l blocks in each row if and only if the constructed graph supports an all-shortest-path ($\max\{k,l\}+1$)-IRS (($\max\{k,l\}+2$)-SIRS). Thus, the transformation can start from an instance of Flammini's NP-complete problem in [5] to prove the NP-completeness of all-shortest-path k-IRS (and its variants) for general (but not constant) integer k.

The results of this paper clearly imply that the optimization problem of determining the minimal k such that a given network supports an all-shortest-path k-IRS (or its variants) is NP-hard. They also imply that we cannot in polynomial time approximate the compactness of IRS (SIRS) within a ratio of less than $4/3$ ($5/4$), unless $P = \text{NP}$.

For the global compactness, we have strengthened the NP-completeness results of [5] to cover the unweighted graph and the linear cases. Note that, while no approximation algorithms have been proposed to approximate the edge compactness within constant ratios (to the best of our knowledge), a 2.25-approximation algorithm for the global compactness of IRS and a 1.5-approximation algorithm for the global compactness of SIRS have been designed [5].

Some remaining open problems: What is the time complexity when $k = 1, 2$ ($k = 2, 3$) for all-shortest-path k-IRS (k-SIRS)?

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