Optimal Gossiping in Square 2D Meshes ⋆

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Abstract

Gossiping is the communication problem in which each node has a unique message to be transmitted to every other node. The nodes exchange their message by packets. A solution to the problem is judged by how many rounds of packet sending it requires. In this paper, we consider the version of the problem in which small-size packets each carrying exactly one message are used. The nodes of the target meshes are assumed to be all-port (a node's incident edges can all be active at the same time); and their edges are either half-duplex or full-duplex, which are also known as the H* model and the F* model, respectively. We study the class of 2D square meshes. Soch and Tvrdik (SIROCCO'97, pp. 253–265; Tech. rep. DC-97-04, Dept. of CS&E, Czech Technical University) have obtained optimal algorithms for the F* model (for square or nonsquare meshes). Lau and Zhang (IEEE Trans. on Parallel and Distributed Systems, Vol. 13, No. 4, pp. 349–358, 2002) have obtained fast algorithms for the H* model. We present optimal algorithms for both models, with the interesting property that they route their messages along the same paths and in the same order, i.e., for any edge $\{u, v\}$, the i-th message from u to v under either model is the same message.

Key words: Gossiping, meshes, information dissemination, networks, algorithms.

1 Introduction

Gossiping, also known as total exchange or all-to-all (nonpersonalized) broadcast, is a communication problem in which each processor (or node) has a unique message to be transmitted to every other processor. The gossiping

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process advances by rounds (or timesteps) in a lock-step fashion. In each round, a packet containing one or more messages can only travel across one edge (link). Because of its rich communication pattern, gossiping is a useful benchmark for evaluating the communication capability of an interconnection structure. The gossiping problem has been studied extensively during the last two decades; a summary of the results can be found in [9,11–13]. Gossiping as an embedded operation is needed in many real computations, such as matrix multiplication, LU-factorization, Householder transformation, direct N-body computation, global processor synchronization, and load balancing. Juurlink et al. [14] cited two specific applications that require gossiping as a subroutine: splitter-based sorting [15] and parallel block predictor-corrector methods to solve ordinary differential equations [22].

Krumme et al. have suggested that the gossiping problem can be studied under four different communication models, which have different restrictions on the use of the links, as well as the ability of a node in handling its incident links [16]. The four models are (1) the full-duplex, all-port model, (2) the full-duplex, one-port model, (3) the half-duplex, all-port model, and (4) the half-duplex, one-port model, which can be identified by the labels F*, F1, H*, H1, respectively. A full-duplex link allows both ends to send/receive a message at the same time; a half-duplex link allows only one end to do so at a time. In the one-port mode, only one of the incident links of a node may be active at a time; all the incident links may be active at the same time in the all-port mode. The four models, therefore, form a spectrum, with F* being the strongest in communication capability and H1 the weakest. Krumme et al. studied the problem for a number of well-known topologies under the H1 model [16] and for the hypercube under both the H* and the H1 model [17].

Bermond et al. [5] have added another dimension to the problem. They suggested that a packet carrying messages cannot be of infinite size, which a great majority of previous work had assumed. In reality, indeed, a packet's delay is somewhat dependent on its contents, especially in tightly coupled multiprocessors. They studied the gossiping problem under this hypothesis and under the F1 and F* models, deriving results for the complete graph, hypercube, cycle, torus, Cayley, star, and path [3,4]. Bagchi et al. have considered the same, but under the H1 model [1,2]. Soch and Tvrdik studied meshes and tori under the F* model [23–25].

We can use the parameter p to denote the size of a packet: $p = 1$ means that a packet can carry up to one message, $p = 2$ two messages, and so on. In this paper, we consider only the case of $p = 1$ and focus on the 2D mesh which is an important communication fabric for modern parallel machines. Parallel machines that use 2D meshes include the MIR J-Machine [7], the Symult 2010 [26], and Intel Touchstone [20]. Under the restriction of bounded packets size, the following are the existing results for meshes.

F* model: For $p = 1$, Soch and Tvrdik obtained the optimal result, $\lceil (mn - 1)/2 \rceil$, for the $m \times n$ mesh [24,25].

F1 model: For $p = 1$, Bermond et al. gave an algorithm that solves the problem for $m \times n$ meshes in $2mn - 3 - max\{m, n\}$ steps, m and n odd [5].

H* model: For $p = 1$, Fujita and Yamashita gave an algorithm that can solve the problem for $n \times n$ (square) mesh in $n(n+1)/2 + \lfloor (3n-5)/2 \rfloor$ steps [10]. Based on the optimal gossiping in path given in [19], Lau and Zhang [18] improved this to $n(n+1)/2 + \lfloor (n-1)/2 \rfloor$.

H1 model: For any value of p, Bagchi et al. derived the result $2mn/p + O(m+n)$, for the $m \times n$ mesh [2].

In this paper, we assume the all-port mode; and consider the case of $p = 1$, and hence the terms packet and message become synonymous. Both the F* and the H* model are realistic models for router implementation. There are pros and cons to operating a link in half- or full-duplex mode (see the discussion in [8]). One well-known example of H* router is the Network Design Frame [6]. The C104 router for transputer [21] is an example of the F* model.

We present optimal algorithms for $n \times n$ square meshes under the H* and the F* models with $p = 1$. The algorithms gossip along shortest paths and can be switched easily between the half- and full-duplex modes.

2 Preliminaries

An $n \times n$ 2D mesh $M_{n \times n} = (V, E)$ has $|V| = n^2$ vertices and $|E| = 2n(n-1)$ edges, and consists of n rows and n columns. Fig. 2.1 gives an example—$M_{5 \times 5}$.

2.1 Lower Bounds

Define $g_{F*}(N)$ and $g_{H*}(N)$ to be the times (in rounds) required to complete a gossip for the interconnection network N with $p = 1$, for the F* and the H* model respectively. Existing lower bounds for square meshes are as follows.

Lemma 2.1 $g_{H*}(M_{n \times n}) \geqslant (n+1)/2$; $g_{F*}(M_{n \times n}) \geqslant \lceil (n^2 - 1)/2 \rceil$.

PROOF. [10,18,24] A message traveling across an edge is viewed as one move, the total number of moves in $M_{n \times n}$ is equal to $n(n^2 - 1)$. Some edge has to accommodate at least $\lceil n(n^2-1)/(2n(n-1)) \rceil = n(n+1)/2$ moves. In the H* model each edge at a time can carry at most one move.

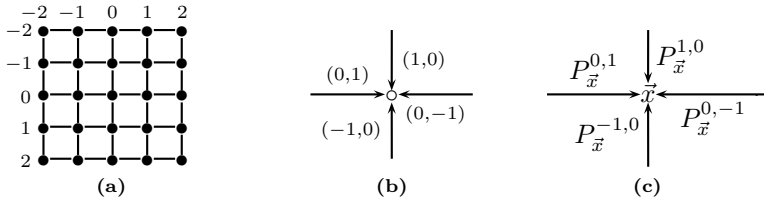

Fig. 2.1. **(a)** $M_{5\times 5}$. **(b)** directions. **(c)** gathering at $\vec{x}$.

For the F* model, consider a corner node of $M_{n\times n}$. It collects n^2-1 messages via the two incident edges; the collection needs at least $\lceil (n^2-1)/2 \rceil$ rounds. □

2.2 Vectors

As shown in Fig. 2.1(a), for $M_{n\times n}$, where $n = 2k+1$ is odd, we number the rows (resp. the columns) with $-k, -k+1, \ldots, -1, 0, 1, \ldots k-1, k$ from top to bottom (from left to right); if n is even ($2k$), we omit row 0 and column 0. A node (vertex) in V is denoted by (x_1, x_2), where x_1 and x_2 are the node's row and column number, respectively. In this paper, the label (x_1, x_2) is also used for the message which originates (at round 0) at the node (x_1, x_2).

For message movements, we use $(0,0)$, $(-1,0)$, $(1,0)$, $(0,-1)$ and $(0,1)$ to refer to the zero, up, down, left and right directions, respectively. See Fig. 2.1(b). Thus, a node, or a message, or a direction is a two dimensional vector. Throughout this paper, we use lower-case letters with an arrow on top for vectors, e.g., $\vec{x} = (x_1, x_2)$, $\vec{y} = (y_1, y_2)$, $\vec{m} = (m_1, m_2)$. Specifically, $\vec{a} = (a_1, a_2)$ is for a direction (may be zero), $\vec{0}$ for $(0,0)$, $\vec{k}$ for (k,k). Classical relations and operations on vectors are employed, e.g., $\vec{x} \leqslant \vec{y} \iff x_1 \leqslant y_1 \bigwedge x_2 \leqslant y_2$, $\vec{x} \pm \vec{y} = (x_1 \pm y_1,\ x_2 \pm y_2)$, $i\vec{x} = (ix_1,\ ix_2)$, and $\vec{x}\vec{y} = x_1y_1 + x_2y_2$.

When n is even, we do not expect $\vec{x} \pm \vec{y}$ to have 0 coordinates because of the omission of row 0 and column 0. We use the sign function

$$\delta(i) = \begin{cases} 1 & i>0, \\ 0 & i=0, \\ -1 & i<0 \end{cases}$$

to define new integer operators $\oplus$ and $\ominus$:

$$i \oplus j = \begin{cases} i+j & \delta(i+j)=\delta(i), \\ i+j+\delta(j) & otherwise \end{cases} \text{ and } i \ominus j = \begin{cases} i-j & \delta(i-j)=\delta(i), \\ i-j-\delta(j) & otherwise \end{cases}$$

and change the + and − operators on vectors to

$$\vec{x}+\vec{y}=\begin{cases}(x_1+x_2,\ y_1+y_2)\ n\ odd,\\ (x_1\oplus x_2,\ y_1\oplus y_2)\ n\ even\end{cases}\quad\text{and}\quad \vec{x}-\vec{y}=\begin{cases}(x_1-x_2,\ y_1-y_2)\ n\ odd,\\ (x_1\ominus x_2,\ y_1\ominus y_2)\ n\ even.\end{cases}$$

Although the change destroys the commutative law for operators + and − in the even n case, it assures us that $\{\vec{x},\vec{x}-\vec{a}\}$ is always an edge of the mesh if $\vec{a}$ is nonzero and both $\vec{x}$ and $\vec{x}-\vec{a}$ are within the mesh. Obviously, $\vec{x}$ is a mesh node only if $-\vec{k}\leqslant\vec{x}\leqslant\vec{k}$.

2.3 Gossiping Schemes

Initially, at round 0, each node holds its own message that must eventually be transmitted to all the other nodes. A gossiping algorithm begins its work at round 1, prescribing for every node which messages from which neighbors the node should collect, and at which rounds; it ends when every node has received all the messages. Let $P_{\vec{x}}^{\vec{a}}$ be the set of all the messages the node $\vec{x}$ collected from node $\vec{x}-\vec{a}$ (via the edge $\{\vec{x},\ \vec{x}-\vec{a}\}$; see Fig. 2.1(c)). Assuming duplication freedom, i.e., a message will not reach the same node more than once (the redundant transportations can be viewed as being idle), we denote by $R_{\vec{x}}(\vec{m})$ the round the message $\vec{m}$ is collected by node $\vec{x}$. Let $P=\left\{P_{\vec{x}}^{\vec{a}}\ \middle|\ \vec{x}\in V,\ \vec{a}\text{ is a direction}\right\}$, called the *message gathering*, and $R=\left\{R_{\vec{x}}\ \middle|\ \vec{x}\in V\right\}$, called the *round assignment*, then a gossiping algorithm is completely represented by (P,R), which we call a *gossiping scheme.*

Definition 2.2 *An F* Gossiping Scheme* (GS) *on mesh* $M_{n\times n}=(V,E)$ *is a pair* (P,R) *such that* (P,R) *is*

(1) complete: $P_{\vec{x}}^{0,0}\cup P_{\vec{x}}^{0,1}\cup P_{\vec{x}}^{1,0}\cup P_{\vec{x}}^{0,-1}\cup P_{\vec{x}}^{-1,0}=\{\vec{m}\ \big|\ \vec{m}\in V\}$*, i.e., every message* $\vec{m}$ *will eventually reach every node* $\vec{x}$*;*
(2) initialized: $P_{\vec{x}}^{\vec{0}}=\{\vec{x}\}$ *and* $R_{\vec{x}}(\vec{x})=0$*, i.e., message* $\vec{x}$ *is viewed as coming to the node* $\vec{x}$ *along zero direction in zero round;*
(3) 1-bounded: $\vec{m},\vec{m}'\in P_{\vec{x}}^{\vec{a}},\vec{m}\neq\vec{m}'\Longrightarrow R_{\vec{x}}(\vec{m})\neq R_{\vec{x}}(\vec{m}')$*, i.e.,* $p=1$*;*
(4) duplication free: $\vec{a}\neq\vec{b}\Longrightarrow P_{\vec{x}}^{\vec{a}}\cap P_{\vec{x}}^{\vec{b}}=\phi$*;*
(5) precedence constrained: $\vec{a}\neq\vec{0},\vec{m}\in P_{\vec{x}}^{\vec{a}}\Longrightarrow R_{\vec{x}}(\vec{m})>R_{\vec{x}-\vec{a}}(\vec{m})$*, i.e., a message leaving a node must have first arrived at the node.*

The H* model gossiping scheme has a further restriction.

Definition 2.3 *An F* GS* (P,R) *is also an H* GS if all links are* half-duplex: $\vec{m}\in P_{\vec{x}}^{\vec{a}},\vec{m}'\in P_{\vec{x}-\vec{a}}^{-\vec{a}}\Longrightarrow R_{\vec{x}}(\vec{m})\neq R_{\vec{x}-\vec{a}}(\vec{m}')$.

In the above definitions, $P_{\vec{x}}^{\vec{a}}$ is assumed empty if $\vec{x}-\vec{a}$ is outside the mesh. Note that the definition is from the viewpoint of gathering, not broadcasting, of messages. Our gossiping scheme is therefore not the same as the standard ones in the literature (see for example [13]).

The quality of a GS (P,R) is measured by $\max_{\vec{x},\vec{m}\in V} R_{\vec{x}}(\vec{m})$, the time it requires to complete the gossip. If this number is no more than r, the gossiping scheme is termed an r-GS.

2.4 Message Arriving Order

Definition 2.4 *For an edge $\{\vec{x}-\vec{a},\vec{x}\}$, if message $\vec{m}$ is the i-th one coming to the node $\vec{x}$ via the edge, then the $S_{\vec{x}}(\vec{m})=i$ is the* arriving order *of $\vec{m}$ at $\vec{x}$. Specifically, we define $S_{\vec{x}}(\vec{x})=0$.*

Example 2.5 *Suppose that node $\vec{x}-\vec{a}$ sends totally 5 messages to node $\vec{x}$ in the whole gossiping process, and the messages arrive at $\vec{x}$ in rounds 2,3,5,7,8, respectively, then their arriving orders at $\vec{x}$ are 1,2,3,4,5, respectively.*

By the definition, we have $S_{\vec{x}}(\vec{m}) \leqslant R_{\vec{x}}(\vec{m})$ and $S_{\vec{x}}(\vec{m}) \leqslant \left|P_{\vec{x}}^{\vec{a}}\right|$ for $\vec{m}\in P_{\vec{x}}^{\vec{a}}$.

2.5 Rotational Symmetry

Turning a direction (a_1,a_2) anticlockwise by 90° gives the direction $(-a_2,a_1)$, and rotating a square mesh anticlockwise by 90° makes the node (x_1,x_2) overlap with the node $(-x_2,x_1)$ in its original position. We use the operator Δ for rotation of vectors: $\Delta(x_1,x_2)=(-x_2,x_1)$, and the abbreviation Δ^i for $\underbrace{\Delta\Delta\ldots\Delta}_{i\ times}$. Clearly $\Delta^0\vec{x}=\vec{x}$, and $\Delta^i\vec{x}=-\Delta^{i\pm 2}\vec{x}$.

Definition 2.6 *A message gathering P is* rotational symmetric *if $\vec{m}\in P_{\vec{x}}^{\vec{a}} \Longleftrightarrow \Delta\vec{m}\in P_{\Delta\vec{x}}^{\Delta\vec{a}}$; a round assignment R is* rotational symmetric *if $R_{\vec{x}}(\vec{m}) = R_{\Delta\vec{x}}(\Delta\vec{m})$; a GS (P,R) is* rotational symmetric *if both P and R are.*

We design rotational symmetric schemes. So we need to consider only one quadrant of the mesh, say $P_{\vec{x}}^{\vec{a}}$ and $R_{\vec{x}}(\vec{m})$ for $-\vec{k}\leqslant\vec{x}\leqslant\vec{0}$.

2.6 Edge Orientation Schedule

In the H* model, we need to decide which direction an edge should take in a given round. Intuitively, an edge should not stick to the same direction for

too long a time; otherwise, some sending node might be exhausted. Success is more likely if we flip the edge's direction as frequently as possible, especially in the first few rounds during which the nodes have not accumulated too many messages. With this intuition, our strategy is to let an edge start with the majority direction (along which more messages travel), and reverse the direction at each round until messages of the minority direction have all been transported. This strategy can be formulated as $R_{\vec{x}}(\vec{m}) = \gamma^{\vec{a}}_{\vec{x}}\left(S_{\vec{x}}(\vec{m})\right)$ for $\vec{m} \in P^{\vec{a}}_{\vec{x}}$, where

$$\gamma^{\vec{a}}_{\vec{x}}(s) = \begin{cases} 2s & \left|P^{\vec{a}}_{\vec{x}}\right| \leqslant \left|P^{-\vec{a}}_{\vec{x}-\vec{a}}\right|, \\ 2s-1 & \left|P^{\vec{a}}_{\vec{x}}\right| > \left|P^{-\vec{a}}_{\vec{x}-\vec{a}}\right|,\ s \leqslant \left|P^{-\vec{a}}_{\vec{x}-\vec{a}}\right|, \\ s + \left|P^{-\vec{a}}_{\vec{x}-\vec{a}}\right| & \left|P^{\vec{a}}_{\vec{x}}\right| > \left|P^{-\vec{a}}_{\vec{x}-\vec{a}}\right|,\ s > \left|P^{-\vec{a}}_{\vec{x}-\vec{a}}\right|. \end{cases} \tag{2.1}$$

To understand the above, consider an edge $\{\vec{x}, \vec{x}-\vec{a}\}$. There are $\left|P^{\vec{a}}_{\vec{x}}\right|$ messages from $\vec{x}-\vec{a}$ to $\vec{x}$ and $\left|P^{-\vec{a}}_{\vec{x}-\vec{a}}\right|$ messages from $\vec{x}$ to $\vec{x}-\vec{a}$. If $\left|P^{\vec{a}}_{\vec{x}}\right| < \left|P^{-\vec{a}}_{\vec{x}-\vec{a}}\right|$, $\vec{x}$ would receive from $\vec{x}-\vec{a}$ at even rounds $2, 4, \ldots, 2\left|P^{\vec{a}}_{\vec{x}}\right|$; otherwise, $\vec{x}$ would receive from $\vec{x}-\vec{a}$ at odd rounds $1,\ 3, \ldots,\ 2\left|P^{-\vec{a}}_{\vec{x}-\vec{a}}\right| - 1$ and all of the rounds from $2\left|P^{-\vec{a}}_{\vec{x}-\vec{a}}\right| + 1$ to $\left|P^{-\vec{a}}_{\vec{x}-\vec{a}}\right| + \left|P^{\vec{a}}_{\vec{x}}\right|$.

Note that the schedule is applicable only if $\left|P^{\vec{a}}_{\vec{x}}\right| \neq \left|P^{-\vec{a}}_{\vec{x}-\vec{a}}\right|$.

2.7 Results

For square mesh $M_{n\times n}$, we give a message gathering P and a round assignment R such that (P, R) is an optimal H* GS, and the message arriving order of (P, R), S, when used as a round assignment, makes (P, S) an optimal F* GS. Given these results, we note the interesting relationship between the algorithms respectively for the H* and the F* model:

- they have identical routing paths (sharing the same P), and
- one scheme's message arriving order is another scheme's round assignment.

That is, over the same edge and in the same direction, the two algorithms send the same messages in the same order. So, the same algorithm can easily switch between the half- and the full-duplex mode. Moreover, both schemes gossip along shortest paths.

The schemes for the odd n case are presented in Section 3, and that for the even n case in Section 4. The proofs are in Sections 5 and 6. Section 7 concludes the paper with some discussion.

3 Odd n

Throughout this section, we assume $n = 2k + 1$ with $k \geqslant 1$. We will present an H* $(n(n+1)/2)$-GS (P, R) and an F* $((n^2-1)/2)$-GS (P, S) for $M_{n\times n}$, where S is the message arriving order of (P, R). P is presented in Section 3.1, and R, as well as S, in Section 3.2.

Note that in this section, for a mesh node $\vec{x}$ and a vertical (horizontal) direction $\vec{a} \neq \vec{0}$, $\vec{x} - \vec{a}\vec{x}\vec{a}$ is the node at row 0 (column 0) in the same column (row) with $\vec{x}$; node $\vec{x} - (k + \vec{a}\vec{x})\vec{a}$ is on the mesh boundary that $\vec{a}$ is moving away from, and $\vec{x} + (k - \vec{a}\vec{x})\vec{a}$ on the boundary $\vec{a}$ is heading towards.

3.1 P

Our final aim is to design a message gathering P that supports both an optimal H* GS and an optimal F* GS. We first relax the problem to that of a design for $|P^{\vec{a}}_{\vec{x}}|$, and then present the solution for $P^{\vec{a}}_{\vec{x}}$. Some requirements for $\left|P^{\vec{a}}_{\vec{x}}\right|$ are as follows.

Req 1. For P to be a qualified message gathering, $\left|P^{\vec{a}}_{\vec{x}}\right|$ must have the initialization property, the duplication freeness, and the completeness property:

$$\left|P^{\vec{0}}_{\vec{x}}\right| = 1, \quad \sum_{0\leqslant i\leqslant 3} \left|P^{\Delta^i\vec{a}}_{\vec{x}}\right| = n^2 - 1 \text{ for } \vec{a} \neq \vec{0}.$$

Req 2. An H* GS attempting to match the lower bound $n(n+1)/2$ must keep every edge busy at all times, i.e., each edge has to transport $n(n+1)/2$ messages (both directions together):

$$\left|P^{\vec{a}}_{\vec{x}}\right| + \left|P^{-\vec{a}}_{\vec{x}-\vec{a}}\right| = \frac{n(n+1)}{2} \quad \text{for } \vec{a} \neq \vec{0}.$$

Req 3. For the optimality of the F* GS, the usability of the edge orientation schedule (2.1), and rotational symmetry, $\left|P^{\vec{a}}_{\vec{x}}\right|$ must be such that

$$\left|P^{\vec{a}}_{\vec{x}}\right| \leqslant \left\lceil \frac{n^2-1}{2} \right\rceil, \quad \left|P^{\vec{a}}_{\vec{x}}\right| \neq \left|P^{-\vec{a}}_{\vec{x}-\vec{a}}\right| \text{ for } \vec{a} \neq \vec{0}, \quad \left|P^{\vec{a}}_{\vec{x}}\right| = \left|P^{\Delta\vec{a}}_{\Delta\vec{x}}\right|.$$

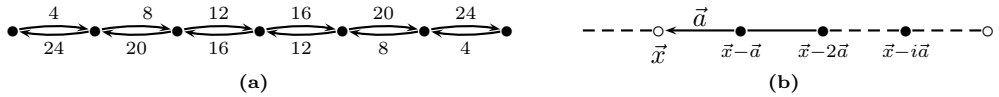

Fig. 3.1. **(a)** $\left|P_{\vec{x}}^{\vec{a}}\right|$ for $\vec{x}$ in a row or column of $M_{7\times 7}$. **(b)** $I_{\vec{x}}^{\vec{a}}(i)$, the solid dots.

Note that these requirements are also for the even n case. For odd n, taking the boundary condition ($\vec{a}\vec{x} = -k$) into account, we choose

$$\left|P_{\vec{x}}^{\vec{a}}\right| = \begin{cases} 0 & \vec{a}\vec{x}=-k, \\ 1 & \vec{a}=\vec{0}, \\ \left|P_{\vec{x}-\vec{a}}^{\vec{a}}\right| + k + 1 & otherwise \end{cases} = \begin{cases} 1 & \vec{a}=\vec{0}, \\ (k+\vec{a}\vec{x})(k+1) & \vec{a}\neq\vec{0} \end{cases} \tag{3.1}$$

which satisfies Req. 1–3. Fig. 3.1(a) shows an example for $\left|P_{\vec{x}}^{\vec{a}}\right|$ in $M_{7\times 7}$. The solution, (3.1), is not only rotational symmetric, but also symmetric with respect to flipping over the diagonals or the zero-th row or the zero-th column. Some particular properties of $\left|P_{\vec{x}}^{\vec{a}}\right|$ are, for $\vec{a} \neq \vec{0}$,

$$\vec{a}\vec{x} \leqslant 0 \Longrightarrow \left|P_{\vec{x}}^{\vec{a}}\right| < \left|P_{\vec{x}-\vec{a}}^{-\vec{a}}\right|, \quad and \quad \vec{a}\vec{x} > 0 \Longrightarrow \left|P_{\vec{x}}^{\vec{a}}\right| > \left|P_{\vec{x}-\vec{a}}^{-\vec{a}}\right|. \tag{3.2}$$

Now we consider the design of $P_{\vec{x}}^{\vec{a}}$ under the restriction of (3.3). A simple idea, suggested by (3.1), is for $-k < \vec{a}\vec{x} \leqslant k$ and $\vec{a} \neq \vec{0}$ to assume $P_{\vec{x}}^{\vec{a}} = P_{\vec{x}-\vec{a}}^{\vec{0}} \cup P_{\vec{x}-\vec{a}}^{\vec{a}} \cup \{\text{some } k \text{ other messages}\}$. The "some k other messages" must be of $P_{\vec{x}-\vec{a}}^{\Delta\vec{a}} \cup P_{\vec{x}-\vec{a}}^{-\Delta\vec{a}}$, and cannot be from $P_{\vec{x}-\vec{a}}^{-\vec{a}}$ (otherwise, the message $\vec{m} \in P_{\vec{x}}^{\vec{a}} \cap P_{\vec{x}-\vec{a}}^{-\vec{a}}$ would visit $\vec{x}$ or $\vec{x}-\vec{a}$ more than once, and thus not duplication free). Denote the set of these k messages by $Q_{\vec{x}}^{\vec{a}}$; then we have, for $\vec{a} \neq \vec{0}$,

$$P_{\vec{x}}^{\vec{a}} = P_{\vec{x}-\vec{a}}^{\vec{0}} \cup P_{\vec{x}-\vec{a}}^{\vec{a}} \cup Q_{\vec{x}}^{\vec{a}}$$

where $\left|Q_{\vec{x}}^{\vec{a}}\right| = k$ and $Q_{\vec{x}}^{\vec{a}} \subseteq P_{\vec{x}-\vec{a}}^{\Delta\vec{a}} \cup P_{\vec{x}-\vec{a}}^{-\Delta\vec{a}}$. Nevertheless, this simple idea of letting a message go straight ahead until reaching the mesh boundary is not applicable to an optimal H* GS.

A message $\vec{m}$ is considered *dead* at node $\vec{x}$ if it travels to $\vec{x}$ in round $R_{\vec{x}}(\vec{m}) = n(n+1)/2$. If there is a dead message $\vec{m} \in P_{\vec{x}-\vec{a}}^{\vec{a}}$ at node $\vec{x}-\vec{a}$, we cannot let $P_{\vec{x}-\vec{a}}^{\vec{a}} \subseteq P_{\vec{x}}^{\vec{a}}$; otherwise, the dead message $\vec{m}$ would leave for $\vec{x}$ in a round larger than $n(n+1)/2$, and the H* GS based on P would not match the lower bound.

According to the edge orientation schedule (2.1), for every edge $\{\vec{x},\ \vec{x}-\vec{a}\}$, the last message passing through it passes in round $\left|P_{\vec{x}}^{\vec{a}}\right| + \left|P_{\vec{x}-\vec{a}}^{-\vec{a}}\right| = n(n+1)/2$, is dead. If $\left|P_{\vec{x}}^{\vec{a}}\right| > \left|P_{\vec{x}-\vec{a}}^{-\vec{a}}\right|$, this message is in $P_{\vec{x}}^{\vec{a}}$ (and dead at $\vec{x}$); otherwise, it is in $P_{\vec{x}-\vec{a}}^{-\vec{a}}$ (and dead at $\vec{x}-\vec{a}$). By (3.2), $\left|P_{\vec{x}}^{\vec{a}}\right| > \left|P_{\vec{x}-\vec{a}}^{-\vec{a}}\right| \iff \vec{a}\vec{x} > 0$. So, we claim the following.

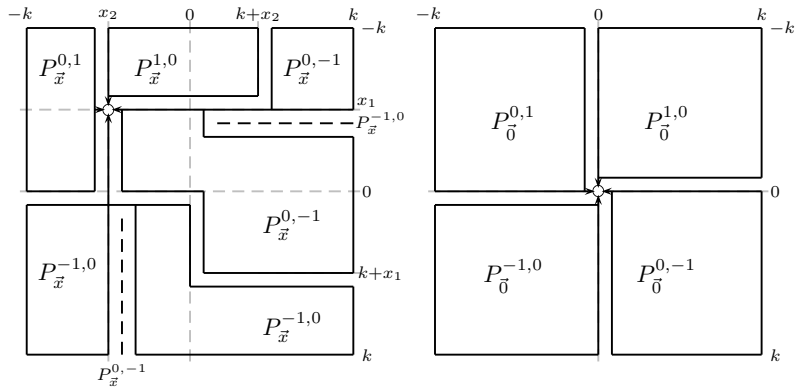

Fig. 3.2. $P^{\vec{a}}_{\vec{x}}$ for $-\vec{k} \leqslant \vec{x} \leqslant (0,-1)$ (the l.h.s) or $\vec{x} = \vec{0}$ (the r.h.s). For other $\vec{x}$'s in the mesh, simply apply a rotation.

Assertion 1 *$P^{\vec{a}}_{\vec{x}}$ has a dead message (at $\vec{x}$) if and only if $\vec{a}\vec{x} > 0$; and $\vec{m} \in P^{\vec{a}}_{\vec{x}}$ is dead at $\vec{x}$ if and only if $\vec{a}\vec{x} > 0$ and $S_{\vec{x}}(\vec{m}) = \left|P^{\vec{a}}_{\vec{x}}\right|$.*

Hence, if $\vec{a}\vec{x} > 1$, there exists a dead message $\vec{m}$ in $P^{\vec{a}}_{\vec{x}-\vec{a}}$ which is dead at $\vec{x}-\vec{a}$, and we must not let $P^{\vec{a}}_{\vec{x}-\vec{a}} \subseteq P^{\vec{a}}_{\vec{x}}$. Let $G^{\vec{a}}_{\vec{x}} = P^{\vec{a}}_{\vec{x}-\vec{a}} - P^{\vec{a}}_{\vec{x}}$, the messages from $P^{\vec{a}}_{\vec{x}-\vec{a}}$ which are given up by $P^{\vec{a}}_{\vec{x}}$; then to preserve (3.1), $\vec{x}$ must receive the same number of messages (as the number given up) from $P^{\Delta\vec{a}}_{\vec{x}-\vec{a}} \cup P^{-\Delta\vec{a}}_{\vec{x}-\vec{a}}$ in addition to $Q^{\vec{a}}_{\vec{x}}$; denoting these *makeup* messages by $M^{\vec{a}}_{\vec{x}}$, our design for $P^{\vec{a}}_{\vec{x}}$, taking the boundary condition ($\vec{a} = \vec{0}$ or $\vec{a}\vec{k} = -k$) into account, is

$$P^{\vec{a}}_{\vec{x}} = \begin{cases} \phi & \vec{a}\vec{x}=-k, \\ \{\vec{x}\} & \vec{a}=\vec{0}, \\ P^{\vec{0}}_{\vec{x}-\vec{a}} \cup \left(P^{\vec{a}}_{\vec{x}-\vec{a}} - G^{\vec{a}}_{\vec{x}}\right) \cup Q^{\vec{a}}_{\vec{x}} \cup M^{\vec{a}}_{\vec{x}} & otherwise; \end{cases} \tag{3.3}$$

where by the above discussion,

$$Q^{\vec{a}}_{\vec{x}} \cup M^{\vec{a}}_{\vec{x}} \subseteq P^{\pm\Delta\vec{a}}_{\vec{x}-\vec{a}},\ M^{\vec{a}}_{\vec{x}} \cap Q^{\vec{a}}_{\vec{x}} = \phi,\ \left|Q^{\vec{a}}_{\vec{x}}\right| = k,\ \left|M^{\vec{a}}_{\vec{x}}\right| = \left|G^{\vec{a}}_{\vec{x}}\right|, \text{ and } G^{\vec{a}}_{\vec{x}} \subseteq P^{\vec{a}}_{\vec{x}-\vec{a}}.$$

Given (3.3), we need only to determine $Q^{\vec{a}}_{\vec{x}}$, $M^{\vec{a}}_{\vec{x}}$ and $G^{\vec{a}}_{\vec{x}}$. The final design is illustrated in Fig. 3.2, and an example is given in Fig. 3.3. There are two dashed lines in Fig. 3.2, each covers $k-1$ messages. The vertical dashed line represents $M^{0,-1}_{\vec{x}}$, and also $G^{0,-1}_{\vec{x}+(0,-1)}$, or $G^{-1,0}_{\vec{x}-(-1,0)\vec{x}(-1,0)}$; similarly, the horizontal dashed line represents $M^{-1,0}_{\vec{x}}$, and also $G^{-1,0}_{\vec{x}+(-1,0)}$, or $G^{0,-1}_{\vec{x}-(0,-1)\vec{x}(0,-1)+(0,-1)}$. It can be seen that we have tried to shape $P^{\vec{a}}_{\vec{x}}$ to be as "regular" as possible, and let $Q^{\vec{a}}_{\vec{x}}$, $M^{\vec{a}}_{\vec{x}}$ and $G^{\vec{a}}_{\vec{x}}$ be lines. To describe these lines, for $0 \leqslant i \leqslant k + \vec{a}\vec{x}$, we define

$$I^{\vec{a}}_{\vec{x}}(i) = \left\{\vec{x} - j\vec{a} \;\middle|\; {}_{1\leqslant j\leqslant i}\right\}$$

to be the set of i messages immediately next to node $\vec{x}$ along the direction $-\vec{a}$, as shown in Fig. 3.1(b). Then, $Q^{\vec{a}}_{\vec{x}}$, $M^{\vec{a}}_{\vec{x}}$ and $G^{\vec{a}}_{\vec{x}}$ can be formulated as follows.

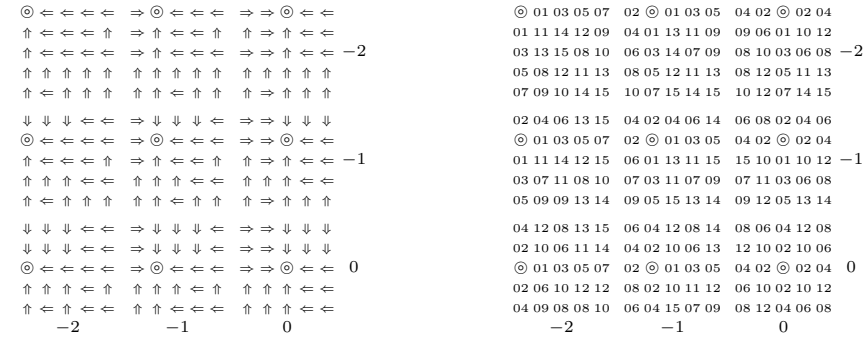

Fig. 3.3. $P^{\vec{a}}_{\vec{x}}$ (the l.h.s) and $R_{\vec{x}}(\vec{m})$ (the r.h.s). Each 5×5 submatrix is for one node $\vec{x}$ of $M_{5\times 5}$ with $(-2,-2)\leqslant\vec{x}\leqslant(0,0)$, giving the directions (in the l.h.s) and the H* model rounds (in the r.h.s) by which messages would come to the node $\vec{x}$. The situation for $\vec{x}$ in other quadrants is similar, by symmetry.

$$Q^{\vec{a}}_{\vec{x}}=\begin{cases} I^{-\Delta\vec{a}}_{\vec{x}-\vec{a}}\,(k) & \vec{a}\vec{x}\leqslant 0,\ \Delta\vec{a}\vec{x}\leqslant 0,\\ I^{-\Delta\vec{a}}_{\vec{x}-\vec{a}}\,(k-\Delta\vec{a}\vec{x})\cup I^{\Delta\vec{a}}_{\vec{x}-\vec{a}}\,(\Delta\vec{a}\vec{x}) & \vec{a}\vec{x}\leqslant 0,\ \Delta\vec{a}\vec{x}>0,\\ I^{-\Delta\vec{a}}_{\vec{x}-\vec{a}}\,(-\Delta\vec{a}\vec{x})\cup I^{\Delta\vec{a}}_{\vec{x}-(k+1)\vec{a}}\,(k+\Delta\vec{a}\vec{x}) & \vec{a}\vec{x}>0,\ \Delta\vec{a}\vec{x}\leqslant 0,\\ I^{\Delta\vec{a}}_{\vec{x}-(k+1)\vec{a}-\Delta\vec{a}\vec{x}\Delta\vec{a}}\,(k) & \vec{a}\vec{x}>0,\ \Delta\vec{a}\vec{x}>0; \end{cases} \tag{3.4}$$

$$M^{\vec{a}}_{\vec{x}}=\begin{cases} I^{-\Delta\vec{a}}_{\vec{x}-\vec{a}+(1-\Delta\vec{a}\vec{x})\Delta\vec{a}}\,(k-1) & \vec{a}\vec{x}>0,\ \Delta\vec{a}\vec{x}\leqslant 0,\\ I^{\Delta\vec{a}}_{\vec{x}-\vec{a}-(1+\Delta\vec{a}\vec{x})\Delta\vec{a}}\,(k-1) & \vec{a}\vec{x}\geqslant 0,\ \Delta\vec{a}\vec{x}>0,\\ \phi & otherwise; \end{cases} \tag{3.5}$$

$$G^{\vec{a}}_{\vec{x}}=P^{\vec{a}}_{\vec{x}-\vec{a}}-P^{\vec{a}}_{\vec{x}}=\begin{cases} I^{\vec{a}}_{\vec{x}-2\vec{a}+\Delta\vec{a}}\,(k-1) & \vec{a}\vec{x}=1,\ \Delta\vec{a}\vec{x}\leqslant 0,\\ I^{\vec{a}}_{\vec{x}-\vec{a}-\Delta\vec{a}}\,(k-1) & \vec{a}\vec{x}=0,\ \Delta\vec{a}\vec{x}>0,\\ M^{\vec{a}}_{\vec{x}-\vec{a}} & otherwise. \end{cases} \tag{3.6}$$

Note that $G^{\vec{a}}_{\vec{x}}=M^{\vec{a}}_{\vec{x}}=\phi$ if $\vec{a}\vec{x}<0$. For $\vec{a}\vec{x}>1$, $G^{\vec{a}}_{\vec{x}}=M^{\vec{a}}_{\vec{x}-\vec{a}}$ and $\left|G^{\vec{a}}_{\vec{x}}\right|=\left|M^{\vec{a}}_{\vec{x}-\vec{a}}\right|=k-1$, where the former means that the messages made up by $P^{\vec{a}}_{\vec{x}-\vec{a}}$ will be given up by $P^{\vec{a}}_{\vec{x}}$, and $k-1$ in the later is due to the fact that for a nonzero direction $\vec{a}$ and $-k<\vec{a}\vec{x}\leqslant k$ there are totally $k-1$ dead messages in the $P^{\vec{a}}_{\vec{x}-\vec{a}}$'s (no dead message in $P^{\vec{a}}_{\vec{x}-\vec{a}}$ for $\vec{a}\vec{x}\leqslant 1$, and one for each $1<\vec{a}\vec{x}\leqslant k$). It can be checked that the design satisfies (3.1) and therefore Req. 1–3 and (3.2).

3.2 R and S

Our aim is to design an R such that both R and its message arriving order S are qualified round assignments (respectively for the H* and the F* model).

The edge orientation schedule (2.1) provides a way to design R. We need only to design S, i.e., to label the messages of $P^{\vec{a}}_{\vec{x}}$ from 1 to $|P^{\vec{a}}_{\vec{x}}|$, provided that S and the induced R (by $R_{\vec{x}}(\vec{m}) = \gamma^{\vec{a}}_{\vec{x}}\Big(S_{\vec{x}}(\vec{m})\Big)$) are precedence constrained.

A natural idea in the design of the order for the messages in $P^{\vec{a}}_{\vec{x}}$ is "first come to $\vec{x}-\vec{a}$, first go to $\vec{x}$". However, for the precedence constraint, this idea has to be integrated with the consideration regarding the future journeys of the messages.

The messages of $P^{\vec{a}}_{\vec{x}}$ could be differentiated by the movements they will have after reaching $\vec{x}$. While leaving $\vec{x}$ for $\vec{x}+\vec{a}$ to continue their journey in direction $\vec{a}$, those making a turn at $\vec{x}$ and leaving for $\vec{x}-\Delta\vec{a}$ or $\vec{x}+\Delta\vec{a}$, or both, are called *productive messages*. The other messages of $P^{\vec{a}}_{\vec{x}}$ will only go straight ahead in direction $\vec{a}$, which can be further divided according to whether or not they will reach the node $\vec{x}+(k-\vec{a}\vec{x})\vec{a}$, the mesh boundary they are heading towards; if yes, they are *ordinary*, otherwise, *abortive*.

It is reasonable to *speed up* the productive messages (let them arrive earlier at node $\vec{x}$) and *slow down* the abortive ones (to delay their arrival at $\vec{x}$). So our idea in deciding $S_{\vec{x}}(\vec{m})$ for $\vec{m} \in P^{\vec{a}}_{\vec{x}}$ is a mixture of the "first come first go" principle and the "speedup/slowdown" principle. To formalize this idea, we need to identify the abortive and productive messages of $P^{\vec{a}}_{\vec{x}}$. From (3.3)–(3.6), it can be inferred that the abortive messages and the productive messages of $P^{\vec{a}}_{\vec{x}}$ should be $\bigcup_{0\leqslant i<k-\vec{a}\vec{x}}\Big(P^{\vec{a}}_{\vec{x}} \cap G^{\vec{a}}_{\vec{x}+i\vec{a}}\Big)$ and $P^{\vec{a}}_{\vec{x}} \cap \Big(P^{-\Delta\vec{a}}_{\vec{x}-\Delta\vec{a}} \cup P^{\Delta\vec{a}}_{\vec{x}+\Delta\vec{a}}\Big)$, respectively. For simplicity, we let the abortive message set be

$$A^{\vec{a}}_{\vec{x}} = \begin{cases} I^{\vec{a}}_{\vec{x}+\Delta\vec{a}}\,(k+\vec{a}\vec{x}) & \vec{a}\vec{x}<0,\ \Delta\vec{a}\vec{x}\leqslant 0, \\ I^{\vec{a}}_{\vec{x}-\Delta\vec{a}}\,(k+\vec{a}\vec{x}) & \vec{a}\vec{x}<0,\ \Delta\vec{a}\vec{x}>0, \\ I^{\vec{a}}_{\vec{x}-\vec{a}+\Delta\vec{a}}\,(k-1) & \vec{a}\vec{x}=0,\ \Delta\vec{a}\vec{x}\leqslant 0, \\ M^{\vec{a}}_{\vec{x}} & otherwise, \end{cases}$$

which is a superset of $\bigcup_{0\leqslant i<k-\vec{a}\vec{x}}\Big(P^{\vec{a}}_{\vec{x}} \cap G^{\vec{a}}_{\vec{x}+i\vec{a}}\Big)$. Similarly, we let the productive set be $W^{\vec{a}}_{\vec{x}} = \Psi^{\vec{a}}_{\vec{x}} \cup \Gamma^{\vec{a}}_{\vec{x}}$, a superset of $P^{\vec{a}}_{\vec{x}} \cap \Big(P^{-\Delta\vec{a}}_{\vec{x}-\Delta\vec{a}} \cup P^{\Delta\vec{a}}_{\vec{x}+\Delta\vec{a}}\Big)$, where

$$\Psi^{\vec{a}}_{\vec{x}} = I^{\vec{a}}_{\vec{x}}\,(k+\vec{a}\vec{x}) \quad \text{and} \quad \Gamma^{\vec{a}}_{\vec{x}} = \begin{cases} I^{\vec{a}}_{\vec{x}+k\Delta\vec{a}}\,(k+\vec{a}\vec{x}) & \vec{a}\vec{x}\leqslant 0,\ \Delta\vec{a}\vec{x}\leqslant 0, \\ I^{\vec{a}}_{\vec{x}-\vec{a}\vec{x}\vec{a}+k\Delta\vec{a}}\,(k) & \vec{a}\vec{x}>0,\ \Delta\vec{a}\vec{x}\leqslant 0, \\ \phi & otherwise. \end{cases}$$

Then, the H* round assignment R, and its message arriving order S which also acts as the F* round assignment, can be decided by the following procedure.

Procedure $RS(\vec{x},\vec{a})$**:** To compute $S_{\vec{x}}(\vec{m})$ and $R_{\vec{x}}(\vec{m})$ for $\vec{m} \in P^{\vec{a}}_{\vec{x}}$.

1. If $\vec{a} = \vec{0}$ then for $\vec{m} \in P^{\vec{a}}_{\vec{x}}$, set $S_{\vec{x}}(\vec{m}) = 0$, $R_{\vec{x}}(\vec{m}) = 0$, end.
2. $B = \phi$. B is to record the used order numbers.
3. For $C = \Psi^{\vec{a}}_{\vec{x}}, \Gamma^{\vec{a}}_{\vec{x}}, P^{\vec{a}}_{\vec{x}-\vec{a}} - \left(W^{\vec{a}}_{\vec{x}} \cup A^{\vec{a}}_{\vec{x}}\right), Q^{\vec{a}}_{\vec{x}} - W^{\vec{a}}_{\vec{x}}, A^{\vec{a}}_{\vec{x}}$, respectively, while $C \neq \phi$ repeat the following three steps.
   a. Let $\vec{m} \in C$ such that $S_{\vec{x}-\vec{a}}(\vec{m}) = \min\left\{S_{\vec{x}-\vec{a}}(\vec{m}) \;\middle|\; \vec{m} \in C\right\}$.
   b. Let s be the smallest integer not in B such that $s > S_{\vec{x}-\vec{a}}(\vec{m})$ and $\gamma^{\vec{a}}_{\vec{x}}(s) > R_{\vec{x}-\vec{a}}(\vec{m})$.
   c. Set $S_{\vec{x}}(\vec{m}) = s$, $R_{\vec{x}}(\vec{m}) = \gamma^{\vec{a}}_{\vec{x}}(s)$, $B = B \cup \{s\}$, and $C = C - \{\vec{m}\}$.

In the procedure, Step 3 is to do 3.a–3.c cyclically first for $C = \Psi^{\vec{a}}_{\vec{x}}$, and then for $C = \Gamma^{\vec{a}}_{\vec{x}}$, etc., reflecting the "speed up the productive messages and slow down the abortive messages" idea. For each set, Steps 3.a and 3.b implement the "first come to $\vec{x}-\vec{a}$, first go to $\vec{x}$" idea. Although the procedure is recursive (the decision for $S_{\vec{x}}(\vec{m})$ depends on the value of $S_{\vec{x}-\vec{a}}(\vec{m})$), it can be implemented with dynamic programming techniques and complete the computation within polynomial time of n.

Fig. 3.3 shows the values of $R_{\vec{x}}(\vec{m})$ in $M_{5\times 5}$. We conclude with the following theorem the proof of which is in Section 5.

Theorem 3.1 *(P, S) is an F^* $((n^2-1)/2)$-GS and (P, R) is an H^* $(n(n+1)/2)$-GS, both of which route messages along the shortest paths in $M_{n\times n}$, where $n > 1$ is odd.*

4 Even n

The same ideas for odd n apply equally well to even n $(= 2k)$. As in the odd n case, the two gossiping schemes are rotational symmetric and based on the same P. The $P^{\vec{a}}_{\vec{x}}$ for even n takes the same form (3.3) as that for odd n:

$$P^{\vec{a}}_{\vec{x}} = \begin{cases} \phi & \vec{a}\vec{x}{=}{-}k, \\ \{\vec{x}\} & \vec{a}{=}\vec{0}, \\ P^{\vec{0}}_{\vec{x}-\vec{a}} \cup \left(P^{\vec{a}}_{\vec{x}-\vec{a}} - G^{\vec{a}}_{\vec{x}}\right) \cup Q^{\vec{a}}_{\vec{x}} \cup M^{\vec{a}}_{\vec{x}} & otherwise; \end{cases}$$

of which $Q^{\vec{a}}_{\vec{x}}$, $M^{\vec{a}}_{\vec{x}}$, and $G^{\vec{a}}_{\vec{x}}$ need some small changes, as follows.

$$Q^{\vec{a}}_{\vec{x}} = \begin{cases} I^{-\Delta\vec{a}}_{\vec{x}-\vec{a}}\,(k-1) & \vec{a}\vec{x}{<}0,\ \Delta\vec{a}\vec{x}{<}0, \\ I^{-\Delta\vec{a}}_{\vec{x}-\vec{a}}\,(k-\Delta\vec{a}\vec{x}) \cup I^{\Delta\vec{a}}_{\vec{x}-\vec{a}}\,(\Delta\vec{a}\vec{x}) & \vec{a}\vec{x}{<}0,\ \Delta\vec{a}\vec{x}{>}0, \\ I^{-\Delta\vec{a}}_{\vec{x}-\vec{a}}\,(-\Delta\vec{a}\vec{x}) \cup I^{\Delta\vec{a}}_{\vec{x}-k\vec{a}}\,(k+\Delta\vec{a}\vec{x}) & \vec{a}\vec{x}{>}0,\ \Delta\vec{a}\vec{x}{<}0, \\ I^{\Delta\vec{a}}_{\vec{x}-k\vec{a}-\Delta\vec{a}\vec{x}\Delta\vec{a}}\,(k-1) & \vec{a}\vec{x}{>}0,\ \Delta\vec{a}\vec{x}{>}0; \end{cases} \tag{4.1}$$

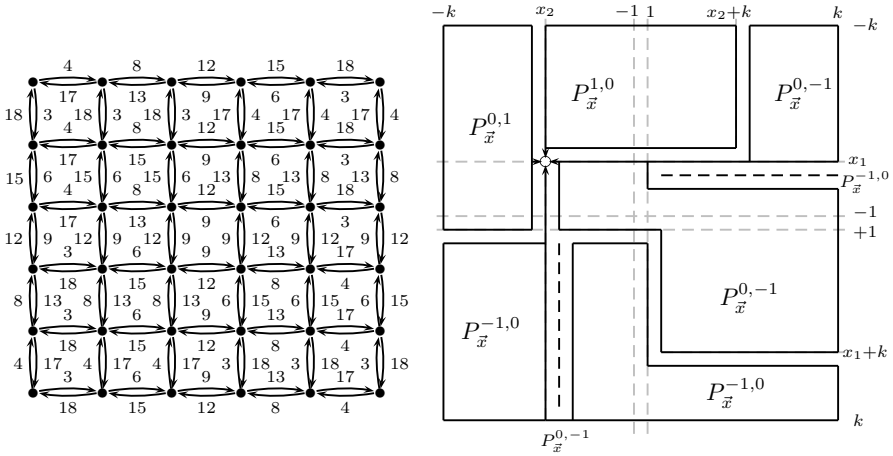

Fig. 4.1. $\left|P^{\vec{a}}_{\vec{x}}\right|$ in $M_{6\times 6}$ (l.h.s) and $P^{\vec{a}}_{\vec{x}}$ at $\vec{x}$ for $-\vec{k} \leqslant \vec{x} < \vec{0}$ (r.h.s).

$$M^{\vec{a}}_{\vec{x}} = \begin{cases} I^{\Delta\vec{a}}_{\vec{x}-\vec{a}-\Delta\vec{a}\vec{x}\Delta\vec{a}}(k-2) & \vec{a}\vec{x}{>}0,\ \Delta\vec{a}\vec{x}{<}0, \\ I^{\Delta\vec{a}}_{\vec{x}-\vec{a}-\Delta\vec{a}\vec{x}\Delta\vec{a}}(k-1) & \vec{a}\vec{x}{\geqslant}0,\ \Delta\vec{a}\vec{x}{>}0, \\ \phi & otherwise; \end{cases} \quad G^{\vec{a}}_{\vec{x}} = \begin{cases} I^{\vec{a}}_{\vec{x}+\Delta\vec{a}}(k-1) & \vec{a}\vec{x}{=}1,\ \Delta\vec{a}\vec{x}{<}0, \\ I^{\vec{a}}_{\vec{x}-\vec{a}-\Delta\vec{a}}(k-2) & \vec{a}\vec{x}{=}1,\ \Delta\vec{a}\vec{x}{>}0, \\ M^{\vec{a}}_{\vec{x}-\vec{a}} & otherwise. \end{cases}$$

Note that, for even n, $\vec{x}+\vec{y} = (x_1 \oplus x_2, y_1 \oplus y_2)$, $\vec{x}-\vec{y} = (x_1 \ominus x_2, y_1 \ominus y_2)$, and $I^{\vec{a}}_{\vec{x}}(t)$ remains to be the set of the t messages immediately next to $\vec{x}$ in direction $-\vec{a}$. For $-\vec{k} \leqslant \vec{x} < \vec{0}$, $P^{\vec{a}}_{\vec{x}}$ is illustrated in the r.h.s of Fig. 4.1.

Clearly, P is initialized, complete, duplication free, and in accordance with Req. 1–3. As illustrated in the l.h.s of Fig. 4.1, an example of $\left|P^{\vec{a}}_{\vec{x}}\right|$ for $M_{6\times 6}$, we have

$$\left|P^{\vec{a}}_{\vec{x}}\right| = \begin{cases} 0 & \vec{a}\vec{x}{=}{-}k, \\ 1 & \vec{a}{=}\vec{0}, \\ \left|P^{\vec{a}}_{\vec{x}-\vec{a}}\right| + k & \vec{a}\vec{x}{\leqslant}1,\ \Delta\vec{a}\vec{x}{\leqslant}0, \\ \left|P^{\vec{a}}_{\vec{x}-\vec{a}}\right| + k + 1 & \vec{a}\vec{x}{\leqslant}1,\ \Delta\vec{a}\vec{x}{>}0, \\ \left|P^{\vec{a}}_{\vec{x}-\vec{a}}\right| + k + 1 & \vec{a}\vec{x}{>}1,\ \Delta\vec{a}\vec{x}{\leqslant}0, \\ \left|P^{\vec{a}}_{\vec{x}-\vec{a}}\right| + k & \vec{a}\vec{x}{>}1,\ \Delta\vec{a}\vec{x}{>}0 \end{cases} = \begin{cases} (k+\vec{a}\vec{x})k & \vec{a}\vec{x}{<}0,\ \Delta\vec{a}\vec{x}{<}0, \\ (k+\vec{a}\vec{x})(k+1) & \vec{a}\vec{x}{<}0,\ \Delta\vec{a}\vec{x}{>}0, \\ (k+\vec{a}\vec{x})k + \vec{a}\vec{x} - k - 1 & \vec{a}\vec{x}{>}0,\ \Delta\vec{a}\vec{x}{<}0, \\ (k+\vec{a}\vec{x})k & \vec{a}\vec{x}{>}0,\ \Delta\vec{a}\vec{x}{>}0. \end{cases}$$

Note that for the even n case, (3.2) no longer holds. We still have $\left|P^{\vec{a}}_{\vec{x}}\right| \neq \left|P^{-\vec{a}}_{\vec{x}-\vec{a}}\right|$ for $\vec{a} \neq \vec{0}$, but now $\left|P^{\vec{a}}_{\vec{x}}\right| < \left|P^{-\vec{a}}_{\vec{x}-\vec{a}}\right| \Longleftrightarrow \vec{a}\vec{x} \leqslant 1 \bigwedge \Delta\vec{a}\vec{x} \leqslant 0 \bigvee \vec{a}\vec{x} < 1 \bigwedge \Delta\vec{a}\vec{x} > 0$. Thus, Assertion 1 should be adapted to even n by replacing all the occurrences of $\vec{a}\vec{x} > 0$ by $\vec{a}\vec{x} > 1 \bigwedge \Delta\vec{a}\vec{x} \leqslant 0 \bigvee \vec{a}\vec{x} > 0 \bigwedge \Delta\vec{a}\vec{x} > 0$ in the claim, or by replacing the occurrences by $\left|P^{\vec{a}}_{\vec{x}}\right| > \left|P^{-\vec{a}}_{\vec{x}-\vec{a}}\right|$, making it generalized for both odd n and even n.

The abortive message set now is

$$A^{\vec{a}}_{\vec{x}} = \begin{cases} I^{\vec{a}}_{\vec{x}+\Delta\vec{a}}\,(k+\vec{a}\vec{x}) & {\scriptstyle \vec{a}\vec{x}<1,\ \Delta\vec{a}\vec{x}\leqslant 0,} \\ I^{\vec{a}}_{\vec{x}-\Delta\vec{a}}\,(k+\vec{a}\vec{x}-1) & {\scriptstyle \vec{a}\vec{x}<1,\ \Delta\vec{a}\vec{x}>0,} \\ M^{\vec{a}}_{\vec{x}} & otherwise. \end{cases}$$

The productive set is still $W^{\vec{a}}_{\vec{x}} = \Psi^{\vec{a}}_{\vec{x}} \cup \Gamma^{\vec{a}}_{\vec{x}}$, but where $\Psi^{\vec{a}}_{\vec{x}}$ and $\Gamma^{\vec{a}}_{\vec{x}}$ are changed to

$$\Psi^{\vec{a}}_{\vec{x}} = \begin{cases} I^{\vec{a}}_{\vec{x}}\,(k+\vec{a}\vec{x}) & {\scriptstyle \vec{a}\vec{x}<0,} \\ I^{\vec{a}}_{\vec{x}}\,(k+\vec{a}\vec{x}-1) & {\scriptstyle \vec{a}\vec{x}>0} \end{cases} \text{ and } \Gamma^{\vec{a}}_{\vec{x}} = \begin{cases} I^{\vec{a}}_{\vec{x}+(k-1)\Delta\vec{a}}\,(k+\vec{a}\vec{x}) & {\scriptstyle \vec{a}\vec{x}<0,\ \Delta\vec{a}\vec{x}\leqslant 1,} \\ I^{\vec{a}}_{\vec{x}-\vec{a}\vec{x}\vec{a}+(k-1)\Delta\vec{a}}\,(k-1) & {\scriptstyle \vec{a}\vec{x}>0,\ \Delta\vec{a}\vec{x}\leqslant 1,} \\ \phi & otherwise. \end{cases}$$

Given $A^{\vec{a}}_{\vec{x}}$ and $W^{\vec{a}}_{\vec{x}}$, Procedure RS can be applied. Let R and S be produced by the procedure, then (P, R) is an optimal H* GS and (P, S) is an optimal F* GS, as claimed in the next theorem and proved in Section 6.

Theorem 4.1 *Let S and R be generated by Procedure RS based on the P designed for the even n case. Then, (P, S) is an F* $\lceil (n^2-1)/2 \rceil$-GS and (P, R) an H* $(n(n+1)/2)$-GS, and the messages of both gossip along the shortest paths in $M_{n\times n}$, where $n > 1$ is even.*

5 The Proof of Theorem 3.1

It can be easily seen that P is complete, initialized, and duplication free. A particular property of P is that

$$\vec{m} \in P^{\vec{a}}_{\vec{x}} \Longrightarrow \vec{a}\vec{m} < \vec{a}\vec{x}\ for\ \vec{a} \neq \vec{0},$$

which limits a node $\vec{x}$ to receive from above (resp. below) only those messages that are above (resp. below) it, and from the left (resp. right) only those on its left (resp. right). This assures us that any scheme based on P would gossip along the shortest paths.

Procedure RS explicitly imposes initialization and the precedence constraint upon S and R, and the 1-boundedness upon S, which implies the 1-boundedness of R because $\gamma^{\vec{a}}_{\vec{x}}(s)$ is monotonously increasing (i.e., $\gamma^{\vec{a}}_{\vec{x}}(s+1) > \gamma^{\vec{a}}_{\vec{x}}(s)$). So (P, S) is an F* GS, and if R is half-duplex, then (P, R) is an H* GS.

According to (2.1), if $S^{\vec{a}}_{\vec{x}}(\vec{m}) \leqslant \left|P^{\vec{a}}_{\vec{x}}\right|$ for $\vec{m} \in P^{\vec{a}}_{\vec{x}}$, the half-duplexity of R as

well as the optimality of S and R, is implied. Therefore, to prove Theorem 3.1, we need only to prove the following.

Proposition 5.1 *Procedure RS produces an S which satisfies $S_{\vec{x}}(\vec{m}) \leqslant \left|P^{\vec{a}}_{\vec{x}}\right|$ for $\vec{m} \in P^{\vec{a}}_{\vec{x}}$.*

To arrive at the above, we need to go through a series of lemmas. We assume $\vec{a} \neq \vec{0}$ since for $\vec{a} = \vec{0}$ the proposition trivially holds. For succinctness, we adopt the following notations.

For a set A of messages, denote $\{S_{\vec{x}}(\vec{m}) \mid \vec{m} \in A\}$ by $S_{\vec{x}}(A)$, and $\{R_{\vec{x}}(\vec{m}) \mid \vec{m} \in A\}$ by $R_{\vec{x}}(A)$. For a set I of integers and a function $f(i)$ on integers, denote $\{f(i) \mid i \in I\}$ by $f(I)$. For two integer sets I and J, we write $I < J$ if there is a bijection $f : I \to J$ such that $i < f(i)$ for each $i \in I$; $I > J$, $I \leqslant J$, and $I \geqslant J$ are defined similarly. For example: $\{3, 1, 6\} < \{5, 2, 7\} \leqslant \{5, 2, 8\}$.

The integer set in this paper is intended for order numbers or round numbers, and so we allow it to contain duplications. For example: $S_{\vec{x}}(A)$ may be $\{2, 3, 3, 5\}$ if A contains messages traveling to $\vec{x}$ through different edges. This change affects the union operation of two integer sets; for example, $\{1, 2, 3\} \cup \{2, 3, 4\}$ is in this proof $\{1, 2, 2, 3, 3, 4\}$, not $\{1, 2, 3, 4\}$. We call an integer set *distinct* if it contains integers that are distinct from each other.

We name $F \subseteq P^{\vec{a}}_{\vec{x}}$ a *prior* subset (of $P^{\vec{a}}_{\vec{x}}$) if Procedure RS does not make decision for anyone of $P^{\vec{a}}_{\vec{x}} - F$ until all the messages of F have been decided. Some examples of prior subsets of $P^{\vec{a}}_{\vec{x}}$ are: $\Psi^{\vec{a}}_{\vec{x}}$, $W^{\vec{a}}_{\vec{x}}$, $P^{\vec{a}}_{\vec{x}} - A^{\vec{a}}_{\vec{x}}$, and $P^{\vec{a}}_{\vec{x}}$. Specifically, ϕ is regarded prior. Obviously, given any two prior subsets of $P^{\vec{a}}_{\vec{x}}$, one of them must be a subset of the other.

With these notations, we start on the way to Proposition 5.1 with a simple observation about Procedure RS.

Observation 1 *Given two prior subsets F_1 and F_2 of $P^{\vec{a}}_{\vec{x}}$ and a distinct integer set J, if $J \cap S_{\vec{x}}(F_1) = \phi$, $|J| = |F_2 - F_1|$, $J > S_{\vec{x}-\vec{a}}(F_2 - F_1)$, and $\gamma^{\vec{a}}_{\vec{x}}(J) > R_{\vec{x}-\vec{a}}(F_2 - F_1)$, then Procedure RS makes $S_{\vec{x}}(F_2 - F_1) \leqslant J$.*

5.1 Preparation

To make the observation more usable, we define an integer function:

$$\alpha^{\vec{a}}_{\vec{x}}(s) = \begin{cases} s+1 & \vec{a}\vec{x} \leqslant 0 \text{ or } s \leqslant \left|P^{-\vec{a}}_{\vec{x}-\vec{a}}\right|, \\ 2s - \left|P^{-\vec{a}}_{\vec{x}-\vec{a}}\right| & \vec{a}\vec{x} > 0,\ \left|P^{-\vec{a}}_{\vec{x}-\vec{a}}\right| < s \leqslant \left|P^{-\vec{a}}_{\vec{x}-2\vec{a}}\right|, \\ s+k+2 & \vec{a}\vec{x} > 0,\ s > \left|P^{-\vec{a}}_{\vec{x}-2\vec{a}}\right| \end{cases} \tag{5.1}$$

and claim the following.

Lemma 5.2 *Suppose message* $\vec{m} \in P^{\vec{a}}_{\vec{x}}$ *with* $S_{\vec{x}-\vec{a}}(\vec{m}) = s$, *if* $\vec{a}\vec{x} \leqslant 0$ *or* $s \leqslant \left|P^{-\vec{a}}_{\vec{x}-\vec{a}}\right|$ *or* $\vec{a}\vec{x} > 1$ *and* $\vec{m} \in P^{\vec{0}}_{\vec{x}-\vec{a}} \cup P^{\vec{a}}_{\vec{x}-\vec{a}}$ *then* $\alpha^{\vec{a}}_{\vec{x}}(s)$ *is the smallest number for* $S_{\vec{x}}(\vec{m})$ *such that* $S_{\vec{x}}(\vec{m}) > S_{\vec{x}-\vec{a}}(\vec{m})$ *and* $R_{\vec{x}}(\vec{m}) > R_{\vec{x}-\vec{a}}(\vec{m})$.

PROOF. By the nature of $\alpha^{\vec{a}}_{\vec{x}}$, we have $\alpha^{\vec{a}}_{\vec{x}}(s+1) > \alpha^{\vec{a}}_{\vec{x}}(s) > s$ for any integer s. So we need only to prove that $\gamma^{\vec{a}}_{\vec{x}}\left(\alpha^{\vec{a}}_{\vec{x}}(s)\right) > R_{\vec{x}-\vec{a}}(\vec{m})$ and that either $\alpha^{\vec{a}}_{\vec{x}}(s) = s+1$ or $\gamma^{\vec{a}}_{\vec{x}}\left(\alpha^{\vec{a}}_{\vec{x}}(s)\right) = R_{\vec{x}-\vec{a}}(\vec{m})+1$. Note that for any direction $\vec{b}$, $2s \geqslant \gamma^{\vec{b}}_{\vec{x}}(s)$ and $\gamma^{\vec{b}}_{\vec{x}}(s+1) > \gamma^{\vec{b}}_{\vec{x}}(s) \geqslant s$, of which the former tells us that $R_{\vec{x}-\vec{a}}(\vec{m}) \leqslant 2s$.

If $\vec{a}\vec{x} \leqslant 0$ or $s \leqslant \left|P^{-\vec{a}}_{\vec{x}-\vec{a}}\right|$, then we have $\alpha^{\vec{a}}_{\vec{x}}(s) = s+1$ and

$$\gamma^{\vec{a}}_{\vec{x}}\left(\alpha^{\vec{a}}_{\vec{x}}(s)\right) = \begin{cases} 2(s+1) & |P^{\vec{a}}_{\vec{x}}|<\left|P^{-\vec{a}}_{\vec{x}-\vec{a}}\right|, \\ 2(s+1)-1 & |P^{\vec{a}}_{\vec{x}}|<\left|P^{-\vec{a}}_{\vec{x}-\vec{a}}\right|,\ s<\left|P^{-\vec{a}}_{\vec{x}-\vec{a}}\right|, \\ \left|P^{-\vec{a}}_{\vec{x}-\vec{a}}\right|+s+1 & |P^{\vec{a}}_{\vec{x}}|<\left|P^{-\vec{a}}_{\vec{x}-\vec{a}}\right|, s=\left|P^{-\vec{a}}_{\vec{x}-\vec{a}}\right| \end{cases} > 2s \geqslant R_{\vec{x}-\vec{a}}(\vec{m}).$$

For $\vec{a}\vec{x} > 1$ and $\vec{m} \in P^{\vec{0}}_{\vec{x}-\vec{a}} \cup P^{\vec{a}}_{\vec{x}-\vec{a}}$, we can, by the above, assume $s > \left|P^{-\vec{a}}_{\vec{x}-\vec{a}}\right|$, which means that $\vec{m} \in P^{\vec{a}}_{\vec{x}-\vec{a}}$ and $R_{\vec{x}-\vec{a}}(\vec{m}) = \gamma^{\vec{a}}_{\vec{x}-\vec{a}}(s)$. Then, there are two possible cases to consider: **(1)** $\left|P^{-\vec{a}}_{\vec{x}-\vec{a}}\right| < s \leqslant \left|P^{-\vec{a}}_{\vec{x}-2\vec{a}}\right|$, and **(2)** $\left|P^{-\vec{a}}_{\vec{x}-2\vec{a}}\right| < s$. Note that $\vec{a}\vec{x} > 1$ means that $\left|P^{\vec{a}}_{\vec{x}-\vec{a}}\right| > \left|P^{-\vec{a}}_{\vec{x}-2\vec{a}}\right| > \left|P^{-\vec{a}}_{\vec{x}-\vec{a}}\right|$.

Case (1): $\left|P^{-\vec{a}}_{\vec{x}-\vec{a}}\right| < s \leqslant \left|P^{-\vec{a}}_{\vec{x}-2\vec{a}}\right|$. Then $\alpha^{\vec{a}}_{\vec{x}}(s) = 2s - \left|P^{-\vec{a}}_{\vec{x}-\vec{a}}\right| > s > \left|P^{-\vec{a}}_{\vec{x}-\vec{a}}\right|$ and $\gamma^{\vec{a}}_{\vec{x}}\left(\alpha^{\vec{a}}_{\vec{x}}(s)\right) = \left|P^{-\vec{a}}_{\vec{x}-\vec{a}}\right| + 2s - \left|P^{-\vec{a}}_{\vec{x}-\vec{a}}\right| = 2s = \gamma^{\vec{a}}_{\vec{x}-\vec{a}}(s)+1$.

Case (2): $s > \left|P^{-\vec{a}}_{\vec{x}-2\vec{a}}\right|$. Then $\alpha^{\vec{a}}_{\vec{x}}(s) = s+k+2$ and $\gamma^{\vec{a}}_{\vec{x}}\left(\alpha^{\vec{a}}_{\vec{x}}(s)\right) = \left|P^{-\vec{a}}_{\vec{x}-\vec{a}}\right| + s + k + 2 = \left|P^{-\vec{a}}_{\vec{x}-2\vec{a}}\right| + s + 1 = \gamma^{\vec{a}}_{\vec{x}-\vec{a}}(s)+1$.

In both cases, we have $\gamma^{\vec{a}}_{\vec{x}}\left(\alpha^{\vec{a}}_{\vec{x}}(s)\right) = R_{\vec{x}-\vec{a}}(\vec{m}) + 1$, which completes the proof. □

Note that $\alpha^{\vec{a}}_{\vec{x}}$ is monotonously increasing, i.e., $\alpha^{\vec{a}}_{\vec{x}}(i+1) > \alpha^{\vec{a}}_{\vec{x}}(i)$, and so is $\gamma^{\vec{a}}_{\vec{x}}$. By combining Observation 1 and Lemma 5.2, it is easy to derive the next.

Corollary 5.3 *Given two prior subsets* F_1 *and* F_2 *of* $P^{\vec{a}}_{\vec{x}}$ *and a distinct integer set* J *such that* $\alpha^{\vec{a}}_{\vec{x}}(J) \cap S_{\vec{x}}(F_1) = \phi$, $|J| = |F_2 - F_1|$, *and* $S_{\vec{x}-\vec{a}}(F_2 - F_1) = J$ $(\leqslant J,$ *or* $< J)$, *if one of the following holds, then Procedure RS makes* $S_{\vec{x}}(F_2 - F_1) = \alpha^{\vec{a}}_{\vec{x}}(J)$ $(\leqslant \alpha^{\vec{a}}_{\vec{x}}(J),$ *or* $< \alpha^{\vec{a}}_{\vec{x}}(J))$.

(a). $\vec{a}\vec{x} \leqslant 0$,
(b). $\max J \leqslant \left|P^{-\vec{a}}_{\vec{x}-\vec{a}}\right|$,
(c). $\vec{a}\vec{x} > 1$ *and* $F_2 - F_1 \subseteq P^{\vec{0}}_{\vec{x}-\vec{a}} \cup P^{\vec{a}}_{\vec{x}-\vec{a}}$.

In most cases, the above corollary will be used with $F_1 = \phi$. As an example, note that if $F \subseteq P^{\vec{0}}_{\vec{x}-\vec{a}} \cup P^{\vec{a}}_{\vec{x}-\vec{a}}$, then $S_{\vec{x}-\vec{a}}(F)$ consists of distinct numbers; taking $J = S_{\vec{x}-\vec{a}}(F)$, $F_1 = \phi$ and $F_2 = F$ in the corollary, we derive

Corollary 5.4 $S_{\vec{x}}(F) = \alpha^{\vec{a}}_{\vec{x}}\left(S_{\vec{x}-\vec{a}}(F)\right)$ *if* $\vec{a}\vec{x} \neq 1$ *and* $F \subseteq P^{\vec{0}}_{\vec{x}-\vec{a}} \cup P^{\vec{a}}_{\vec{x}-\vec{a}}$ *is a prior subset of* $P^{\vec{a}}_{\vec{x}}$.

Lemma 5.5 *Procedure RS gives an S such that*

(a). $S_{\vec{x}}\left(\Psi^{\vec{a}}_{\vec{x}}\right) = \alpha^{\vec{a}}_{\vec{x}}\left(S_{\vec{x}-\vec{a}}\left(\Psi^{\vec{a}}_{\vec{x}}\right)\right)$, *and*
(b). $S_{\vec{x}}(\vec{m}) = \alpha^{\vec{a}}_{\vec{x}}\left(S_{\vec{x}-\vec{a}}(\vec{m})\right)$ *for* $\vec{m} \in \Psi^{\vec{a}}_{\vec{x}}$.

PROOF. **(b)** follows easily from **(a)**. Because $\Psi^{\vec{a}}_{\vec{x}} = P^{\vec{0}}_{\vec{x}-\vec{a}} \cup \Psi^{\vec{a}}_{\vec{x}-\vec{a}} \subseteq P^{\vec{0}}_{\vec{x}-\vec{a}} \cup P^{\vec{a}}_{\vec{x}-\vec{a}}$ is a prior subset of $P^{\vec{a}}_{\vec{x}}$, **(a)** holds if $\vec{a}\vec{x} \neq 1$ by Corollary 5.4.

To prove **(a)** for the case $\vec{a}\vec{x} = 1$, based on the fact that $\Psi^{\vec{a}}_{\vec{x}} = \phi$ at $\vec{a}\vec{x} = -k$, and the fact that **(a)** holds for $\vec{a}\vec{x} \leqslant 0$, it is easy to reason inductively on $\vec{a}\vec{x}$ that $S_{\vec{x}}\left(\Psi^{\vec{a}}_{\vec{x}}\right) = \left\{i \;\middle|\; 1 \leqslant i \leqslant k + \vec{a}\vec{x}\right\}$ for $-k \leqslant \vec{a}\vec{x} \leqslant 0$, leading to $S_{\vec{x}-\vec{a}}\left(\Psi^{\vec{a}}_{\vec{x}-\vec{a}}\right) = \left\{i \;\middle|\; 1 \leqslant i \leqslant k\right\}$ for $\vec{a}(\vec{x} - \vec{a}) = 0$ (i.e. $\vec{a}\vec{x} = 1$). Thus at $\vec{a}\vec{x} = 1$, $S_{\vec{x}-\vec{a}}\left(\Psi^{\vec{a}}_{\vec{x}}\right) = S_{\vec{x}-\vec{a}}\left(P^{\vec{0}}_{\vec{x}-\vec{a}}\right) \cup S_{\vec{x}-\vec{a}}\left(\Psi^{\vec{a}}_{\vec{x}-\vec{a}}\right) = \left\{i \;\middle|\; 0 \leqslant i \leqslant k\right\}$ with $\max S_{\vec{x}-\vec{a}}\left(\Psi^{\vec{a}}_{\vec{x}}\right) = k < \left|P^{-\vec{a}}_{\vec{x}-\vec{a}}\right|$. So, **(a)** holds at $\vec{a}\vec{x} = 1$ (according to of Corollary 5.3(b)). □

Lemma 5.6 *For* $0 \leqslant i \leqslant k + \vec{a}\vec{x}$, *Procedure RS guarantees that*

$$S_{\vec{x}}(\vec{x} - i\vec{a}) = \begin{cases} i & \vec{a}\vec{x} < k \text{ or } i \leqslant k+2, \\ 2i - k - 3 & \text{otherwise.} \end{cases} \tag{5.2}$$

PROOF. We prove the lemma inductively on i. (5.2) trivially holds for $i = 0$ since $S_{\vec{x}}(\vec{x}) = 0$. Suppose (5.2) holds for $i-1$ with $0 < i \leqslant k + \vec{a}\vec{x}$. We have (1) $S_{\vec{x}}(\vec{x} - i\vec{a}) = \alpha^{\vec{a}}_{\vec{x}}\left(S_{\vec{x}-\vec{a}}(\vec{x} - i\vec{a})\right)$ (because $\vec{x} - i\vec{a} \in \Psi^{\vec{a}}_{\vec{x}}$ and by Lemma 5.5), and (2) $\vec{a}(\vec{x} - \vec{a}) = \vec{a}\vec{x} - 1 < k$ (note that we always assume implicitly that $|\vec{a}\vec{x}| \leqslant k$; otherwise $\vec{x}$ is outside of the mesh). Thus, by the induction assumption, $S_{\vec{x}-\vec{a}}(\vec{x} - i\vec{a}) = S_{\vec{x}-\vec{a}}\Big((\vec{x} - \vec{a}) - (i-1)\vec{a}\Big) = i - 1$ and $S_{\vec{x}}(\vec{x} - i\vec{a}) = \alpha^{\vec{a}}_{\vec{x}}(i-1)$. If $\vec{a}\vec{x} < k$ then $\left|P^{-\vec{a}}_{\vec{x}-\vec{a}}\right| = (k - \vec{a}\vec{x} + 1)(k+1) \geqslant 2k + 2 > k + \vec{a}\vec{x} > i - 1$; if $\vec{a}\vec{x} = k$ and $i \leqslant k + 2$ then $\left|P^{-\vec{a}}_{\vec{x}-\vec{a}}\right| = k + 1 \geqslant i - 1$. So, if $\vec{a}\vec{x} < k$ or $i \leqslant k + 2$ then $i - 1 \leqslant \left|P^{-\vec{a}}_{\vec{x}-\vec{a}}\right|$, and by (5.1), $\alpha^{\vec{a}}_{\vec{x}}(i-1) = i$. Thus we have

$S_{\vec{x}}(\vec{x}-i\vec{a})=i$ if $\vec{a}\vec{x}<k$ or $i\leqslant k+2$. If $\vec{a}\vec{x}=k$ and $i>k+2$, we can infer that $\left|P^{-\vec{a}}_{\vec{x}-\vec{a}}\right|=k+1<i-1<2(k+1)=\left|P^{-\vec{a}}_{\vec{x}-2\vec{a}}\right|$, and thus $S_{\vec{x}}(\vec{x}-i\vec{a})=\alpha^{\vec{a}}_{\vec{x}}(i-1)=2(i-1)-\left|P^{-\vec{a}}_{\vec{x}-\vec{a}}\right|=2i-k-3$. We can conclude that (5.2) holds for i, and thus completing the proof. □

Lemma 5.7 *The S given by Procedure RS satisfies:*

(a). $S_{\vec{x}}\left(I^{\vec{a}}_{\vec{x}-\vec{a}\vec{x}\vec{a}-\vec{a}}(k-1)\right)\leqslant\left\{k+2i\;\middle|\;{\scriptstyle 1\leqslant i<k}\right\}$ *for* $\vec{a}\vec{x}\geqslant 0$, *and*
(b). $S_{\vec{x}}\left(\Gamma^{\vec{a}}_{\vec{x}}\right)\leqslant\left\{3k+2i-3\;\middle|\;{\scriptstyle 1\leqslant i\leqslant\left|\Gamma^{\vec{a}}_{\vec{x}}\right|}\right\}$.

PROOF. For **(a)**, note that $I^{\vec{a}}_{\vec{x}-\vec{a}\vec{x}\vec{a}-\vec{a}}(k-1)=\left\{\vec{x}-(\vec{a}\vec{x}+1+i)\vec{a}\;\middle|\;{\scriptstyle 1\leqslant i<k}\right\}$, Thus, by Lemma 5.6, if $0\leqslant\vec{a}\vec{x}<k$ then

$$S_{\vec{x}}\left(I^{\vec{a}}_{\vec{x}-\vec{a}\vec{x}\vec{a}-\vec{a}}(k-1)\right)=\left\{\vec{a}\vec{x}+1+i\;\middle|\;{\scriptstyle 1\leqslant i<k}\right\}\leqslant\left\{k+i\;\middle|\;{\scriptstyle 1\leqslant i<k}\right\};$$

if $\vec{a}\vec{x}=k$, then

$$\begin{aligned}S_{\vec{x}}\left(I^{\vec{a}}_{\vec{x}-\vec{a}\vec{x}\vec{a}-\vec{a}}(k-1)\right)&=S_{\vec{x}}\left(\left\{\vec{x}-(k+1+i)\vec{a}\;\middle|\;{\scriptstyle 1\leqslant i<k}\right\}\right)\\&=\{k+2,k+3,k+5,k+7\ldots,3k-3\}\\&\leqslant\left\{k+2i\;\middle|\;{\scriptstyle 1\leqslant i<k}\right\}.\end{aligned}$$

(a) is proved.

(b) trivially holds for $\Delta\vec{a}\vec{x}>0$ because then $\Gamma^{\vec{a}}_{\vec{x}}=\phi$. So for the proof of **(b)** we assume $\Delta\vec{a}\vec{x}\leqslant 0$. Our first aim is to prove

$$S_{\vec{x}}\left(\Gamma^{\vec{a}}_{\vec{x}}\right)=\alpha^{\vec{a}}_{\vec{x}}\left(S_{\vec{x}-\vec{a}}\left(\Gamma^{\vec{a}}_{\vec{x}-\vec{a}}\right)\right),\tag{5.3}$$

which can be reached via

$$S_{\vec{x}}\left(W^{\vec{a}}_{\vec{x}}\right)=\alpha^{\vec{a}}_{\vec{x}}\left(S_{\vec{x}-\vec{a}}\left(W^{\vec{a}}_{\vec{x}}\right)\right)\tag{5.4}$$

because $W^{\vec{a}}_{\vec{x}}=\Psi^{\vec{a}}_{\vec{x}}\cup\Gamma^{\vec{a}}_{\vec{x}}$ and $S_{\vec{x}}\left(\Psi^{\vec{a}}_{\vec{x}}\right)=\alpha^{\vec{a}}_{\vec{x}}\left(S_{\vec{x}-\vec{a}}\left(\Psi^{\vec{a}}_{\vec{x}}\right)\right)$.

(5.4) holds for $\vec{a}\vec{x}>1$ because the prior subset $W^{\vec{a}}_{\vec{x}}=P^{\vec{0}}_{\vec{x}-\vec{a}}\cup W^{\vec{a}}_{\vec{x}-\vec{a}}\subseteq P^{\vec{0}}_{\vec{x}-\vec{a}}\cup P^{\vec{a}}_{\vec{x}-\vec{a}}$ for $\vec{a}\vec{x}>0$. For $-k\leqslant\vec{a}\vec{x}\leqslant 0$, it can be proved by induction on $\vec{a}\vec{x}$ that

$$S_{\vec{x}}\left(W^{\vec{a}}_{\vec{x}}\right)=\alpha^{\vec{a}}_{\vec{x}}\left(S_{\vec{x}-\vec{a}}\left(W^{\vec{a}}_{\vec{x}}\right)\right)=\left\{k+i\;\middle|\;{\scriptstyle 1\leqslant i\leqslant k+\vec{a}\vec{x}}\right\}\cup\left\{i\;\middle|\;{\scriptstyle 1\leqslant i\leqslant k+\vec{a}\vec{x}}\right\},\tag{5.5}$$

which implies 5.4. The induction is easy to perform, based on that 5.5 is trivially true at $\vec{a}\vec{x}=-k$ since then $W^{\vec{a}}_{\vec{x}}=\phi$, and using Corollary 5.3 and that for $-k<\vec{a}\vec{x}\leqslant 0$ (by Lemma 5.6)

$$S_{\vec{x}-\vec{a}}\left(W^{\vec{a}}_{\vec{x}}\right)=S_{\vec{x}-\vec{a}}\left(\{\vec{x}-\vec{a},\ \vec{x}-\vec{a}+k\Delta\vec{a}\}\cup W^{\vec{a}}_{\vec{x}-\vec{a}}\right)=\{0,\ k\}\cup S_{\vec{x}-\vec{a}}\left(W^{\vec{a}}_{\vec{x}-\vec{a}}\right).$$

Specifically, at $\vec{a}(\vec{x}-\vec{a})=0$, the induction gives $S_{\vec{x}-\vec{a}}\left(W^{\vec{a}}_{\vec{x}-\vec{a}}\right)=\left\{i \;\middle|\; 1\leqslant i\leqslant 2k\right\}$. That is, at $\vec{a}\vec{x}=1$, we have

$$S_{\vec{x}-\vec{a}}\left(W^{\vec{a}}_{\vec{x}}\right)=S_{\vec{x}-\vec{a}}\left(P^{\vec{0}}_{\vec{x}-\vec{a}}\right)\cup S_{\vec{x}-\vec{a}}\left(W^{\vec{a}}_{\vec{x}-\vec{a}}\right)=\left\{i \;\middle|\; 0\leqslant i\leqslant 2k\right\}$$

and

$$\max S_{\vec{x}-\vec{a}}\left(W^{\vec{a}}_{\vec{x}}\right)=\left|W^{\vec{a}}_{\vec{x}}\right|=2k\leqslant k(k+1)=(k-\vec{a}\vec{x}+1)(k+1)=\left|P^{-\vec{a}}_{\vec{x}-\vec{a}}\right|.$$

So, by Corollary 5.3(b),

$$S_{\vec{x}}\left(W^{\vec{a}}_{\vec{x}}\right)=\alpha^{\vec{a}}_{\vec{x}}\left(S_{\vec{x}-\vec{a}}\left(W^{\vec{a}}_{\vec{x}}\right)\right)=\left\{i \;\middle|\; 1\leqslant i\leqslant 2k+1\right\}$$

for $\vec{a}\vec{x}=1$. Thus (5.4) holds for $-k\leqslant\vec{a}\vec{x}\leqslant k$, which implies (5.3).

In fact, the proof for (5.5) gives us that for $-k\leqslant\vec{a}\vec{x}\leqslant 0$,

$$S_{\vec{x}}\left(\Gamma^{\vec{a}}_{\vec{x}}\right)=\left\{k+i \;\middle|\; 1\leqslant i\leqslant k+\vec{a}\vec{x}\right\}=\left\{k+i \;\middle|\; 1\leqslant i\leqslant\left|\Gamma^{\vec{a}}_{\vec{x}}\right|\right\}$$

which is no more than $\left\{3k+2i-3 \;\middle|\; 1\leqslant i\leqslant\left|\Gamma^{\vec{a}}_{\vec{x}}\right|\right\}$. So, **(b)** holds for $-k\leqslant\vec{a}\vec{x}\leqslant 0$.

In particular, we have $S_{\vec{x}}\left(\Gamma^{\vec{a}}_{\vec{x}}\right)=\left\{k+i \;\middle|\; 1\leqslant i\leqslant k\right\}$ at $\vec{a}\vec{x}=0$. Based on this, noting that $\Gamma^{\vec{a}}_{\vec{x}}=\Gamma^{\vec{a}}_{\vec{x}-\vec{a}}$ and $\left|\Gamma^{\vec{a}}_{\vec{x}}\right|=k$ for $0<\vec{a}\vec{x}\leqslant k$, and repeatedly applying $\alpha^{\vec{a}}_{\vec{x}}$, we can see that for $0<\vec{a}\vec{x}\leqslant k-2$,

$$S_{\vec{x}}\left(\Gamma^{\vec{a}}_{\vec{x}}\right)=\alpha^{\vec{a}}_{\vec{x}}\left(S_{\vec{x}-\vec{a}}\left(\Gamma^{\vec{a}}_{\vec{x}-\vec{a}}\right)\right)=\left\{k+\vec{a}\vec{x}+i \;\middle|\; 1\leqslant i\leqslant k\right\}. \tag{5.6}$$

At $\vec{a}\vec{x}=k-1$, $\left|P^{-\vec{a}}_{\vec{x}-\vec{a}}\right|=2(k+1)$, $\left|P^{-\vec{a}}_{\vec{x}-2\vec{a}}\right|=3(k+1)$, and (5.6) gives

$$S_{\vec{x}-\vec{a}}\left(\Gamma^{\vec{a}}_{\vec{x}}\right)=\left\{k+\vec{a}(\vec{x}-\vec{a})+i \;\middle|\; 1\leqslant i\leqslant k\right\}=\left\{2k-2+i \;\middle|\; 1\leqslant i\leqslant k\right\}.$$

So for $\vec{a}\vec{x}=k-1$, we have

$$\begin{aligned}S_{\vec{x}}\left(\Gamma^{\vec{a}}_{\vec{x}}\right)&=\alpha^{\vec{a}}_{\vec{x}}\left(\{2k-1,2k\ldots,3k-2\}\right)\\&=\{2k,2k+1,2k+2,2k+3\}\cup\{2k+4,2k+6,\ldots,4k-6\}.\end{aligned}$$

For $\vec{a}\vec{x}=k$, because $\left|P^{-\vec{a}}_{\vec{x}-\vec{a}}\right|=k+1$ and $\left|P^{-\vec{a}}_{\vec{x}-2\vec{a}}\right|=2(k+1)$, we have

$$\begin{aligned}&S_{\vec{x}}\left(\Gamma^{\vec{a}}_{\vec{x}}\right)\\&=\alpha^{\vec{a}}_{\vec{x}}\left(\{2k,2k+1,2k+2,2k+3\}\right)\cup\alpha^{\vec{a}}_{\vec{x}}\left(\{2k+4,2k+6,\ldots,4k-6\}\right)\\&=\{3k-1,3k+1,3k+3,3k+5\}\cup\{3k+6,3k+8,\ldots,5k-4\}.\end{aligned}$$

Each case leads to $S_{\vec{x}}\left(\Gamma^{\vec{a}}_{\vec{x}}\right)\leqslant\left\{3k+2i-3 \;\middle|\; 1\leqslant i\leqslant k\right\}$, completing the proof. □

Lemma 5.8 *In Procedure RS,*
(a). $S_{\vec{x}-\vec{a}}\left(Q_{\vec{x}}^{\vec{a}}\right) \leqslant \left\{ i \;\middle|\; 1 \leqslant i \leqslant \left|Q_{\vec{x}}^{\vec{a}}\right| \right\}$ *for* $-k \leqslant \vec{a}\vec{x} \leqslant 0$,
(b). $S_{\vec{x}-\vec{a}}\left(Q_{\vec{x}}^{\vec{a}}\right) \leqslant \left\{ 3k+2i-3 \;\middle|\; 1 \leqslant i \leqslant k \right\}$ *for* $0 < \vec{a}\vec{x} \leqslant k$,
(c). $S_{\vec{x}-\vec{a}}\left(M_{\vec{x}}^{\vec{a}}\right) \leqslant \left\{ k+2i \;\middle|\; 1 \leqslant i < k \right\}$ *for* $\vec{a}\vec{x} > 0$ *or* $\vec{a}\vec{x} = 0$ *and* $\Delta\vec{a}\vec{x} > 0$.

PROOF. (a) is a straightforward application of Lemma 5.6 because for $\vec{a}\vec{x} \leqslant 0$ we have from (3.4) that

$$Q_{\vec{x}}^{\vec{a}} = \begin{cases} I_{\vec{x}-\vec{a}}^{-\Delta\vec{a}}(k) & {\scriptstyle \Delta\vec{a}\vec{x}\leqslant 0}, \\ I_{\vec{x}-\vec{a}}^{-\Delta\vec{a}}(k-\Delta\vec{a}\vec{x}) \cup I_{\vec{x}-\vec{a}}^{\Delta\vec{a}}(\Delta\vec{a}\vec{x}) & {\scriptstyle \Delta\vec{a}\vec{x}>0}. \end{cases}$$

For $-k \leqslant \vec{a}\vec{x} \leqslant 0$, we have $Q_{\vec{x}}^{\vec{a}} = \begin{cases} I_{\vec{x}-\vec{a}}^{-\Delta\vec{a}}(-\Delta\vec{a}\vec{x}) \cup I_{\vec{x}-(k+1)\vec{a}}^{\Delta\vec{a}}(k+\Delta\vec{a}\vec{x}) & {\scriptstyle \Delta\vec{a}\vec{x}\leqslant 0}, \\ I_{\vec{x}-(k+1)\vec{a}-\Delta\vec{a}\vec{x}\Delta\vec{a}}^{\Delta\vec{a}}(k) & {\scriptstyle \Delta\vec{a}\vec{x}>0} \end{cases}$

$= \begin{cases} I_{\vec{x}-\vec{a}}^{-\Delta\vec{a}}(-\Delta\vec{a}\vec{x}) \cup \Gamma_{\vec{x}-\vec{a}}^{\Delta\vec{a}} & {\scriptstyle \Delta\vec{a}\vec{x}\leqslant 0}, \\ \Gamma_{\vec{x}-\vec{a}}^{\Delta\vec{a}} & {\scriptstyle \Delta\vec{a}\vec{x}>0}. \end{cases}$

So, by Lemma 5.6 and Lemma 5.7(b), if $\Delta\vec{a}\vec{x} \leqslant 0$ then
$S_{\vec{x}-\vec{a}}\left(Q_{\vec{x}}^{\vec{a}}\right) = S_{\vec{x}-\vec{a}}\left(I_{\vec{x}-\vec{a}}^{-\Delta\vec{a}}(-\Delta\vec{a}\vec{x})\right) \cup S_{\vec{x}-\vec{a}}\left(\Gamma_{\vec{x}-\vec{a}}^{\Delta\vec{a}}\right)$
$\leqslant \left\{ i \;\middle|\; 1 \leqslant i \leqslant -\Delta\vec{a}\vec{x} \right\} \cup \left\{ 3k+2i-3 \;\middle|\; 1 \leqslant i \leqslant k+\Delta\vec{a}\vec{x} \right\} \leqslant \left\{ 3k+2i-3 \;\middle|\; 1 \leqslant i \leqslant k \right\}$,
and the same can be derived directly for $\Delta\vec{a}\vec{x} > 0$. **(b)** is proved.

To prove **(c)**, note that for $\vec{a}\vec{x} > 0$ or $\vec{a}\vec{x} = 0$ and $\Delta\vec{a}\vec{x} > 0$, we have from (3.5)

$$M_{\vec{x}}^{\vec{a}} = \begin{cases} I_{\vec{x}-\vec{a}+(1-\Delta\vec{a}\vec{x})\Delta\vec{a}}^{-\Delta\vec{a}}(k-1) & {\scriptstyle \vec{a}\vec{x}>0,\ \Delta\vec{a}\vec{x}\leqslant 0}, \\ I_{\vec{x}-\vec{a}-(1+\Delta\vec{a}\vec{x})\Delta\vec{a}}^{\Delta\vec{a}}(k-1) & {\scriptstyle \vec{a}\vec{x}\geqslant 0,\ \Delta\vec{a}\vec{x}>0}. \end{cases}$$

Then Lemma 5.7(a) gives us

$$\begin{aligned} S_{\vec{x}-\vec{a}}\left(M_{\vec{x}}^{\vec{a}}\right) &= \begin{cases} S_{\vec{x}-\vec{a}}\left(I_{\vec{x}-\vec{a}+(1-\Delta\vec{a}\vec{x})\Delta\vec{a}}^{-\Delta\vec{a}}(k-1)\right) & {\scriptstyle \vec{a}\vec{x}>0,\ \Delta\vec{a}\vec{x}\leqslant 0}, \\ S_{\vec{x}-\vec{a}}\left(I_{\vec{x}-\vec{a}-(1+\Delta\vec{a}\vec{x})\Delta\vec{a}}^{\Delta\vec{a}}(k-1)\right) & {\scriptstyle \vec{a}\vec{x}\geqslant 0,\ \Delta\vec{a}\vec{x}>0}. \end{cases} \\ &= \begin{cases} S_{\vec{x}-\vec{a}}\left(I_{(\vec{x}-\vec{a}-(-\Delta\vec{a})\vec{x}(-\Delta\vec{a})-(-\Delta\vec{a})}^{(-\Delta\vec{a})}(k-1)\right) & {\scriptstyle \vec{a}\vec{x}>0,\ \Delta\vec{a}\vec{x}\leqslant 0}, \\ S_{\vec{x}-\vec{a}}\left(I_{\vec{x}-\vec{a}-\Delta\vec{a}\vec{x}\Delta\vec{a}-R\vec{a}}^{\Delta\vec{a}}(k-1)\right) & {\scriptstyle \vec{a}\vec{x}\geqslant 0,\ \Delta\vec{a}\vec{x}>0}. \end{cases} \\ &\leqslant \left\{ k+2i \;\middle|\; 1 \leqslant i < k \right\}, \end{aligned}$$

which is **(c)**. □

Finally, we end this preparation with a companion of Lemma 5.8.

Lemma 5.9 *In Procedure RS,*
(a). $R_{\vec{x}-\vec{a}}\left(Q^{\vec{a}}_{\vec{x}}\right) \leqslant \left\{2i \;\middle|\; 1 \leqslant i \leqslant k\right\}$ *for* $-k \leqslant \vec{a}\vec{x} \leqslant 0$,
(b). $R_{\vec{x}-\vec{a}}\left(Q^{\vec{a}}_{\vec{x}}\right) \leqslant \left\{4k+2i-2 \;\middle|\; 1 \leqslant i \leqslant k\right\}$ *for* $0 < \vec{a}\vec{x} \leqslant k$,
(c). $R_{\vec{x}-\vec{a}}\left(M^{\vec{a}}_{\vec{x}}\right) \leqslant \left\{2k+2i+1 \;\middle|\; 1 \leqslant i < k\right\}$ *for* $\vec{a}\vec{x} > 0$ *or* $\vec{a}\vec{x} = 0$ *and* $\Delta\vec{a}\vec{x} > 0$.

PROOF. The way to prove this lemma parallels that for Lemma 5.8. In terms of R, the equation (5.2) in Lemma 5.6 can be stated as

$$R_{\vec{x}}(\vec{x}-i\vec{a}) = \begin{cases} 2i & \vec{a}\vec{x} \leqslant 0, \\ 2i-1 & 0 < \vec{a}\vec{x} < k \text{ or } i \leqslant k+2, \\ 2i-2 & \text{otherwise}, \end{cases}$$

which can be verified with $R_{\vec{x}}(\vec{x}-i\vec{a}) = \gamma_{\vec{x}}\left(S_{\vec{x}}(\vec{x}-i\vec{a})\right)$. Then, the claims in Lemma 5.7 can be written as
(a). $R_{\vec{x}}\left(I^{\vec{a}}_{\vec{x}-\vec{a}\vec{x}\vec{a}-\vec{a}}(k-1)\right) \leqslant \left\{2k+2i+1 \;\middle|\; 1 \leqslant i < k\right\}$ for $\vec{a}\vec{x} \geqslant 0$,
(b). $R_{\vec{x}}\left(\Gamma^{\vec{a}}_{\vec{x}}\right) \leqslant \left\{4k+2i-2 \;\middle|\; 1 \leqslant i \leqslant \left|\Gamma^{\vec{a}}_{\vec{x}}\right|\right\}$.
Their proofs are similar to those for the claims in Lemma 5.7.

Finally, we can arrive at this lemma from the above in the same way as we went from Lemma 5.7 to Lemma 5.8. □

5.2 $\vec{a}\vec{x} \leqslant 0$

Now, we can go to the next theorem, (b) of which implies Proposition 5.1 for the case $\vec{a}\vec{x} \leqslant 0$.

Theorem 5.10 *For* $\vec{a}\vec{x} \leqslant 0$
(a). $S_{\vec{x}}\left(P^{\vec{a}}_{\vec{x}} - A^{\vec{a}}_{\vec{x}}\right) = \left\{i \;\middle|\; 1 \leqslant i \leqslant \left|P^{\vec{a}}_{\vec{x}} - A^{\vec{a}}_{\vec{x}}\right|\right\}$ *for* $\vec{a}\vec{x} > -k+1$,
(b). $S_{\vec{x}}\left(P^{\vec{a}}_{\vec{x}}\right) = \left\{i \;\middle|\; 1 \leqslant i \leqslant \left|P^{\vec{a}}_{\vec{x}}\right|\right\}$ *for* $\vec{a}\vec{x} \geqslant -k$,
(c). $S_{\vec{x}}\left(A^{\vec{a}}_{\vec{x}}\right) = \left\{i \;\middle|\; \left|P^{\vec{a}}_{\vec{x}} - A^{\vec{a}}_{\vec{x}}\right| < i \leqslant \left|P^{\vec{a}}_{\vec{x}}\right|\right\}$ *for* $\vec{a}\vec{x} > -k+1$.

PROOF. If $k=1$ then $-k+1 \geqslant 0$, and **(a)** trivially holds. So the proof for **(a)** assumes $k>1$. For $-k < \vec{a}\vec{x} \leqslant 0$ we have

$$P^{\vec{a}}_{\vec{x}} - A^{\vec{a}}_{\vec{x}} = P^{\vec{0}}_{\vec{x}-\vec{a}} \cup \left(P^{\vec{a}}_{\vec{x}-\vec{a}} - A^{\vec{a}}_{\vec{x}-\vec{a}}\right) \cup Q^{\vec{a}}_{\vec{x}} - \begin{cases} \phi & {\scriptstyle \vec{a}\vec{x}=0,} \\ \{\vec{x}-\vec{a}+\Delta\vec{a}\} & {\scriptstyle \vec{a}\vec{x}<0,\ \Delta\vec{a}\vec{x}\leqslant 0,} \\ \{\vec{x}-\vec{a}-\Delta\vec{a}\} & {\scriptstyle \vec{a}\vec{x}<0,\ \Delta\vec{a}\vec{x}>0.} \end{cases}$$

So, by Lemma 5.6 and Lemma 5.8(a),

$$\begin{aligned} S_{\vec{x}-\vec{a}}\left(P^{\vec{a}}_{\vec{x}} - A^{\vec{a}}_{\vec{x}}\right) &= \{0\} \cup S_{\vec{x}-\vec{a}}\left(P^{\vec{a}}_{\vec{x}-\vec{a}} - A^{\vec{a}}_{\vec{x}-\vec{a}}\right) \cup S_{\vec{x}-\vec{a}}\left(Q^{\vec{a}}_{\vec{x}}\right) - \begin{cases} \phi & {\scriptstyle \vec{a}\vec{x}=0,} \\ \{1\} & {\scriptstyle \vec{a}\vec{x}<0} \end{cases} \\ &\leqslant \{0\} \cup S_{\vec{x}-\vec{a}}\left(P^{\vec{a}}_{\vec{x}-\vec{a}} - A^{\vec{a}}_{\vec{x}-\vec{a}}\right) \cup \begin{cases} \left\{i \;\middle|\; {\scriptstyle 1\leqslant i\leqslant k}\right\} & {\scriptstyle \vec{a}\vec{x}=0,} \\ \left\{i \;\middle|\; {\scriptstyle 1<i\leqslant k}\right\} & {\scriptstyle \vec{a}\vec{x}<0.} \end{cases} \end{aligned} \tag{5.7}$$

At $\vec{a}\vec{x} = -k+2 \leqslant 0$, we have $\vec{a}(\vec{x}-\vec{a}) = -k+1 < 0$, $P^{\vec{a}}_{\vec{x}-2\vec{a}} = \phi$ and $A^{\vec{a}}_{\vec{x}-2\vec{a}} = \phi$, which leads to $S_{\vec{x}-2\vec{a}}\left(P^{\vec{a}}_{\vec{x}-2\vec{a}} - A^{\vec{a}}_{\vec{x}-2\vec{a}}\right) = \phi$. Replacing $\vec{x}$ by $\vec{x}-\vec{a}$ in (5.7) gives us $S_{\vec{x}-2\vec{a}}\left(P^{\vec{a}}_{\vec{x}-\vec{a}} - A^{\vec{a}}_{\vec{x}-\vec{a}}\right) \leqslant \{0\} \cup \left\{i \;\middle|\; {\scriptstyle 1<i\leqslant k}\right\}$. $P^{\vec{a}}_{\vec{x}} - A^{\vec{a}}_{\vec{x}}$ is a prior subset of $P^{\vec{a}}_{\vec{x}}$. So, according to Corollary 5.3(a) (with $F_1 = \phi$ and $F_2 = P^{\vec{a}}_{\vec{x}-\vec{a}} - A^{\vec{a}}_{\vec{x}-\vec{a}}$),

$$S_{\vec{x}-\vec{a}}\left(P^{\vec{a}}_{\vec{x}-\vec{a}} - A^{\vec{a}}_{\vec{x}-\vec{a}}\right) \leqslant \alpha^{\vec{a}}_{\vec{x}-\vec{a}}\left(\{0\} \cup \left\{i \;\middle|\; {\scriptstyle 1<i\leqslant k}\right\}\right) = \{1\} \cup \left\{i \;\middle|\; {\scriptstyle 2<i\leqslant k+1}\right\}.$$

Thus, by (5.7),

$$S_{\vec{x}-\vec{a}}\left(P^{\vec{a}}_{\vec{x}} - A^{\vec{a}}_{\vec{x}}\right) \leqslant \{0\} \cup \{1\} \cup \left\{i \;\middle|\; {\scriptstyle 2<i\leqslant k+1}\right\} \cup \left\{i \;\middle|\; {\scriptstyle 1<i\leqslant k}\right\} \leqslant \left\{i \;\middle|\; {\scriptstyle 0\leqslant i\leqslant 2k}\right\},$$

and then by Corollary 5.3(a) (taking $F_1 = \phi$ and $F_2 = P^{\vec{a}}_{\vec{x}} - A^{\vec{a}}_{\vec{x}}$),

$$S_{\vec{x}}\left(P^{\vec{a}}_{\vec{x}} - A^{\vec{a}}_{\vec{x}}\right) \leqslant \alpha^{\vec{a}}_{\vec{x}}\left(\left\{i \;\middle|\; {\scriptstyle 0\leqslant i\leqslant 2k}\right\}\right) = \left\{i \;\middle|\; {\scriptstyle 1\leqslant i\leqslant 2k+1}\right\} = \left\{i \;\middle|\; {\scriptstyle 1\leqslant i\leqslant \left|P^{\vec{a}}_{\vec{x}} - A^{\vec{a}}_{\vec{x}}\right|}\right\},$$

which implies **(a)** (because $S_{\vec{x}}(B) \geqslant \left\{i \;\middle|\; {\scriptstyle 1\leqslant i\leqslant |B|}\right\}$ for any $B \subseteq P^{\vec{a}}_{\vec{x}}$).

Suppose, for induction on $\vec{a}\vec{x}$, that **(a)** holds for $\vec{a}(\vec{x}-\vec{a}) = \vec{a}\vec{x}-1$. Then $-k+2 < \vec{a}\vec{x} \leqslant 0$, and by (5.7) and the induction assumption, we have

$$S_{\vec{x}-\vec{a}}\left(P^{\vec{a}}_{\vec{x}}-A^{\vec{a}}_{\vec{x}}\right)\leqslant\{0\}\cup\left\{i\;\middle|\;{}_{1<i\leqslant\left|P^{\vec{a}}_{\vec{x}-\vec{a}}-A^{\vec{a}}_{\vec{x}-\vec{a}}\right|}\right\}\cup\begin{cases}\left\{i\;\middle|\;{}_{1\leqslant i\leqslant k}\right\} & \vec{a}\vec{x}=0,\\ \left\{i\;\middle|\;{}_{1<i\leqslant k}\right\} & \vec{a}\vec{x}<0\end{cases}$$
$$\leqslant\begin{cases}\left\{i\;\middle|\;{}_{0\leqslant i\leqslant\left|P^{\vec{a}}_{\vec{x}-\vec{a}}-A^{\vec{a}}_{\vec{x}-\vec{a}}\right|+k}\right\} & \vec{a}\vec{x}=0,\\ \left\{i\;\middle|\;{}_{0\leqslant i\leqslant\left|P^{\vec{a}}_{\vec{x}-\vec{a}}-A^{\vec{a}}_{\vec{x}-\vec{a}}\right|+k-1}\right\} & \vec{a}\vec{x}<0\end{cases}$$
$$=\left\{i\;\middle|\;{}_{0\leqslant i\leqslant\left|P^{\vec{a}}_{\vec{x}}-A^{\vec{a}}_{\vec{x}}\right|-1}\right\}$$

and (by Corollary 5.3(a) with $F_1=\phi$)

$$S_{\vec{x}}\left(P^{\vec{a}}_{\vec{x}}-A^{\vec{a}}_{\vec{x}}\right)\leqslant\alpha^{\vec{a}}_{\vec{x}}\left(\left\{i\;\middle|\;{}_{0\leqslant i\leqslant\left|P^{\vec{a}}_{\vec{x}}\right|-\left|A^{\vec{a}}_{\vec{x}}\right|-1}\right\}\right)=\left\{i\;\middle|\;{}_{1\leqslant i\leqslant\left|P^{\vec{a}}_{\vec{x}}-A^{\vec{a}}_{\vec{x}}\right|}\right\}.$$

Hence **(a)** holds for $-k+1<\vec{a}\vec{x}\leqslant 0$.

Now we prove **(b)** for the case that $\vec{a}\vec{x}<0$ or $\vec{a}\vec{x}=0$ and $\Delta\vec{a}\vec{x}\leqslant 0$. In this case, $P^{\vec{a}}_{\vec{x}}=P^{\vec{0}}_{\vec{x}-\vec{a}}\cup P^{\vec{a}}_{\vec{x}-\vec{a}}\cup Q^{\vec{a}}_{\vec{x}}$, and $S_{\vec{x}-\vec{a}}\left(Q^{\vec{a}}_{\vec{x}}\right)\leqslant\left\{i\;\middle|\;{}_{1\leqslant i\leqslant k}\right\}$ by Lemma 5.8. So,

$$S_{\vec{x}-\vec{a}}\left(P^{\vec{a}}_{\vec{x}}\right)\leqslant\left\{i\;\middle|\;{}_{0\leqslant i\leqslant k}\right\}\cup S_{\vec{x}-\vec{a}}\left(P^{\vec{a}}_{\vec{x}-\vec{a}}\right).$$

Then based on the fact that **(a)** trivially holds at $\vec{a}\vec{x}=-k$ because then $P^{\vec{a}}_{\vec{x}}=\phi$, and noting that $P^{\vec{a}}_{\vec{x}}$ is prior, we can inductively reason that **(b)** holds for $-k\leqslant\vec{a}\vec{x}<0$ or $\vec{a}\vec{x}=0$ and $\Delta\vec{a}\vec{x}\leqslant 0$. We will deal with $\vec{a}\vec{x}=0$ and $\Delta\vec{a}\vec{x}>0$ after the proof of **(c)**.

Combining **(a)** with **(b)** for the case we have proved, **(c)** follows easily for the case $-k-1<\vec{a}\vec{x}<0$ or $\vec{a}\vec{x}=0$ and $\Delta\vec{a}\vec{x}\leqslant 0$. For $\vec{a}\vec{x}=0$ and $\Delta\vec{a}\vec{x}>0$, we have $A^{\vec{a}}_{\vec{x}}=M^{\vec{a}}_{\vec{x}}$, $\left|A^{\vec{a}}_{\vec{x}}\right|=\left|M^{\vec{a}}_{\vec{x}}\right|=k-1$, and (by Lemma 5.8) $S_{\vec{x}-\vec{a}}\left(A^{\vec{a}}_{\vec{x}}\right)\leqslant\left\{k+2i\;\middle|\;{}_{1\leqslant i<k}\right\}$. So $S_{\vec{x}-\vec{a}}\left(A^{\vec{a}}_{\vec{x}}\right)\leqslant\left\{i\;\middle|\;{}_{\left|P^{\vec{a}}_{\vec{x}}-A^{\vec{a}}_{\vec{x}}\right|\leqslant i<\left|P^{\vec{a}}_{\vec{x}}\right|}\right\}$ because for $1\leqslant i<k$,

$$\begin{aligned}\left(\left|P^{\vec{a}}_{\vec{x}}-A^{\vec{a}}_{\vec{x}}\right|-1+i\right)-(k+2i)&=(k(k+1)-(k-1)-1+i)-(k+2i)\\&=k^2-k-i\geqslant k^2-2k+1=(k-1)^2\\&\geqslant 0.\end{aligned}$$

As $P^{\vec{a}}_{\vec{x}}-A^{\vec{a}}_{\vec{x}}$ and $P^{\vec{a}}_{\vec{x}}$ are two prior subsets of $P^{\vec{a}}_{\vec{x}}$, according to Corollary 5.3, we get $S_{\vec{x}}\left(A^{\vec{a}}_{\vec{x}}\right)\leqslant\alpha^{\vec{a}}_{\vec{x}}\left(\left\{i\;\middle|\;{}_{\left|P^{\vec{a}}_{\vec{x}}-A^{\vec{a}}_{\vec{x}}\right|-1\leqslant i<\left|P^{\vec{a}}_{\vec{x}}\right|}\right\}\right)=\left\{i\;\middle|\;{}_{\left|P^{\vec{a}}_{\vec{x}}-A^{\vec{a}}_{\vec{x}}\right|<i\leqslant\left|P^{\vec{a}}_{\vec{x}}\right|}\right\}$ implying **(c)**. Finally, combining this and **(a)**, we get **(b)** for $\vec{a}\vec{x}=0$ and $\Delta\vec{a}\vec{x}>0$. □

5.3 $\vec{a}\vec{x}=1$

In this case, we are led to Proposition 5.1 by (b) of the following theorem.

Theorem 5.11 *The assertions (a), (b), and (c) of Theorem 5.10 also hold at* $\vec{a}\vec{x} = 1$.

PROOF. For $\vec{a}\vec{x} = 1$, i.e. $\vec{a}(\vec{x} - \vec{a}) = 0$, Theorem 5.10(a) gives us that

$$S_{\vec{x}-\vec{a}}\left(P^{\vec{0}}_{\vec{x}-\vec{a}} \cup P^{\vec{a}}_{\vec{x}-\vec{a}} - A^{\vec{a}}_{\vec{x}-\vec{a}}\right) = \left\{i \;\middle|\; 0 \leqslant i \leqslant \left|P^{\vec{a}}_{\vec{x}-\vec{a}} - A^{\vec{a}}_{\vec{x}-\vec{a}}\right|\right\}. \tag{5.8}$$

Note that for $\vec{a}\vec{x} = 1$ we have

(i). $G^{\vec{a}}_{\vec{x}} = A^{\vec{a}}_{\vec{x}-\vec{a}}$, $M^{\vec{a}}_{\vec{x}} = A^{\vec{a}}_{\vec{x}}$, and $P^{\vec{a}}_{\vec{x}} = P^{\vec{0}}_{\vec{x}-\vec{a}} \cup \left(P^{\vec{a}}_{\vec{x}-\vec{a}} - A^{\vec{a}}_{\vec{x}-\vec{a}}\right) \cup Q^{\vec{a}}_{\vec{x}} \cup A^{\vec{a}}_{\vec{x}}$;
(ii). $P^{\vec{0}}_{\vec{x}-\vec{a}} \cup P^{\vec{a}}_{\vec{x}-\vec{a}} - A^{\vec{a}}_{\vec{x}-\vec{a}} = P^{\vec{a}}_{\vec{x}} - \left(Q^{\vec{a}}_{\vec{x}} \cup A^{\vec{a}}_{\vec{x}}\right)$ is a prior subset of $P^{\vec{a}}_{\vec{x}}$;
(iii). $P^{\vec{0}}_{\vec{x}-\vec{a}} \cup P^{\vec{a}}_{\vec{x}-\vec{a}} - A^{\vec{a}}_{\vec{x}-\vec{a}} \cup Q^{\vec{a}}_{\vec{x}} = P^{\vec{a}}_{\vec{x}} - A^{\vec{a}}_{\vec{x}}$ is a prior subset of $P^{\vec{a}}_{\vec{x}}$;
(iv). $\left|P^{\vec{a}}_{\vec{x}-\vec{a}}\right| = \left|P^{-\vec{a}}_{\vec{x}-\vec{a}}\right| = k(k+1) < (k+1)^2 = \left|P^{-\vec{a}}_{\vec{x}-2\vec{a}}\right| = \left|P^{\vec{a}}_{\vec{x}}\right|$.

(iv) in the above gives us $\left|P^{\vec{a}}_{\vec{x}-\vec{a}} - A^{\vec{a}}_{\vec{x}-\vec{a}}\right| > \left|P^{-\vec{a}}_{\vec{x}-\vec{a}}\right|$. From this, (5.8), (ii) in the above, and by Corollary 5.3(b), we get

$$\begin{aligned} S_{\vec{x}}\left(P^{\vec{0}}_{\vec{x}-\vec{a}} \cup P^{\vec{a}}_{\vec{x}-\vec{a}} - A^{\vec{a}}_{\vec{x}-\vec{a}}\right) &= \alpha^{\vec{a}}_{\vec{x}}\left(\left\{i \;\middle|\; 0 \leqslant i \leqslant \left|P^{\vec{a}}_{\vec{x}-\vec{a}} - A^{\vec{a}}_{\vec{x}-\vec{a}}\right|\right\}\right) \\ &= \left\{i \;\middle|\; 1 \leqslant i \leqslant \left|P^{\vec{a}}_{\vec{x}-\vec{a}} - A^{\vec{a}}_{\vec{x}-\vec{a}}\right| + 1\right\}. \end{aligned} \tag{5.9}$$

Now, we prove that for $\vec{a}\vec{x} = 1$,

$$S_{\vec{x}}\left(Q^{\vec{a}}_{\vec{x}}\right) \leqslant \left\{i \;\middle|\; \left|P^{\vec{a}}_{\vec{x}-\vec{a}} - A^{\vec{a}}_{\vec{x}-\vec{a}}\right| + 2 \leqslant i \leqslant \left|P^{\vec{a}}_{\vec{x}-\vec{a}}\right| + 2\right\} \tag{5.10}$$

According to Observation 1, and assuming $F_1 = P^{\vec{0}}_{\vec{x}-\vec{a}} \cup P^{\vec{a}}_{\vec{x}-\vec{a}} - A^{\vec{a}}_{\vec{x}-\vec{a}}$, $F_2 = P^{\vec{0}}_{\vec{x}-\vec{a}} \cup P^{\vec{a}}_{\vec{x}-\vec{a}} - A^{\vec{a}}_{\vec{x}-\vec{a}} \cup Q^{\vec{a}}_{\vec{x}}$, and $J = \left\{i \;\middle|\; \left|P^{\vec{a}}_{\vec{x}-\vec{a}} - A^{\vec{a}}_{\vec{x}-\vec{a}}\right| + 2 \leqslant i \leqslant \left|P^{\vec{a}}_{\vec{x}-\vec{a}}\right| + 2\right\}$, we can get (5.10) if we can justify

$$J > S_{\vec{x}-\vec{a}}\left(Q^{\vec{a}}_{\vec{x}}\right) \text{ and } \gamma^{\vec{a}}_{\vec{x}}(J) > R_{\vec{x}-\vec{a}}\left(Q^{\vec{a}}_{\vec{x}}\right).$$

$J > S_{\vec{x}-\vec{a}}\left(Q^{\vec{a}}_{\vec{x}}\right)$ is a combination of Lemma 5.8(b) and that

$$J = \left\{i \;\middle|\; \left|P^{\vec{a}}_{\vec{x}-\vec{a}} - A^{\vec{a}}_{\vec{x}-\vec{a}}\right| + 2 \leqslant i \leqslant \left|P^{\vec{a}}_{\vec{x}-\vec{a}}\right| + 2\right\} > \left\{3k + 2i - 3 \;\middle|\; 1 \leqslant i \leqslant k\right\},$$

which is true because for $1 \leqslant i \leqslant k$ we have

$$\begin{aligned} &\left(\left|P^{\vec{a}}_{\vec{x}-\vec{a}} - A^{\vec{a}}_{\vec{x}-\vec{a}}\right| + 1 + i\right) - (3k + 2i - 3) \\ &\quad = (k(k+1) - (k-1) + 1 + i) - (3k + 2i - 3) \qquad \textit{by } \left|A^{\vec{a}}_{\vec{x}-\vec{a}}\right| = k-1 \\ &\quad = k^2 - 3k - i + 5 \geqslant k^2 - 4k + 5 = (k-2)^2 + 1 > 0. \end{aligned}$$

$\gamma^{\vec{a}}_{\vec{x}}(J) > R_{\vec{x}-\vec{a}}\left(Q^{\vec{a}}_{\vec{x}}\right)$ because Lemma 5.9(b),

$$\begin{aligned}\gamma^{\vec{a}}_{\vec{x}}(J) &= \gamma^{\vec{a}}_{\vec{x}}\left(\left\{i \;\middle|\; |P^{\vec{a}}_{\vec{x}-\vec{a}}-A^{\vec{a}}_{\vec{x}-\vec{a}}|+2\leqslant i\leqslant |P^{\vec{a}}_{\vec{x}-\vec{a}}|+2\right\}\right)\\ &= \left\{2i-1 \;\middle|\; |P^{\vec{a}}_{\vec{x}-\vec{a}}-A^{\vec{a}}_{\vec{x}-\vec{a}}|+2\leqslant i\leqslant |P^{\vec{a}}_{\vec{x}-\vec{a}}|+1\right\}\cup\left\{2\left|P^{\vec{a}}_{\vec{x}-\vec{a}}\right|+2\right\} \quad \textit{by (iv)}\\ &\geqslant \left\{2i-2 \;\middle|\; |P^{\vec{a}}_{\vec{x}-\vec{a}}-A^{\vec{a}}_{\vec{x}-\vec{a}}|+2\leqslant i\leqslant |P^{\vec{a}}_{\vec{x}-\vec{a}}|+2\right\}\end{aligned}$$

and

$$\begin{aligned}&2\left(\left|P^{\vec{a}}_{\vec{x}-\vec{a}}-A^{\vec{a}}_{\vec{x}-\vec{a}}\right|+1+i\right)-2-(4k+2i-2)\\ &\quad=2(k(k+1)-(k-1)+1+i)-2-(4k+2i-2)\\ &\quad=2k^2-4k+4=k^2+(k-2)^2\geqslant k^2>0.\end{aligned}$$

So, (5.10) holds for $\vec{a}\vec{x}=1$, and moreover, the equality of (5.10) must hold because of (5.9). Then combining this equality with (5.9) results in

$$\begin{aligned}S_{\vec{x}}\left(P^{\vec{a}}_{\vec{x}}-A^{\vec{a}}_{\vec{x}}\right)&=S_{\vec{x}}\left(P^{\vec{0}}_{\vec{x}-\vec{a}}\cup P^{\vec{a}}_{\vec{x}-\vec{a}}-A^{\vec{a}}_{\vec{x}-\vec{a}}\cup Q^{\vec{a}}_{\vec{x}}\right)=\left\{i \;\middle|\; 1\leqslant i\leqslant |P^{\vec{a}}_{\vec{x}-\vec{a}}|+2\right\}\\ &=\left\{i \;\middle|\; 1\leqslant i\leqslant |P^{\vec{a}}_{\vec{x}}|-k+1\right\}=\left\{i \;\middle|\; 1\leqslant i\leqslant |P^{\vec{a}}_{\vec{x}}-A^{\vec{a}}_{\vec{x}}|\right\}.\end{aligned}$$

(a) is proved. $P^{\vec{a}}_{\vec{x}}-A^{\vec{a}}_{\vec{x}}$ and $P^{\vec{a}}_{\vec{x}}$ both are prior subsets of $P^{\vec{a}}_{\vec{x}}$, and the same method for proving (5.10) can be used to prove **(c)** which, combining with **(a)**, will result in **(b)**. So we omit their proofs. □

5.4 $\vec{a}\vec{x}>1$

To arrive at Proposition 5.1 in this case, we need some preparation.

First, for $1\leqslant i<k$ we define $A^{\vec{a}}_{\vec{x}}(i)$ to be the message $\vec{m}$ that is the i-th one of $A^{\vec{a}}_{\vec{x}}$ arriving at node $\vec{x}-\vec{a}$.

Then, we need to define an integer function

$$\beta^{\vec{a}}_{\vec{x}}(i)=\begin{cases}\left|P^{\vec{a}}_{\vec{x}-\vec{a}}\right|+1+i & \vec{a}\vec{x}\leqslant 1,\\ \left|P^{-\vec{a}}_{\vec{x}-\vec{a}}\right|+1+2i & \vec{a}\vec{x}>1\end{cases} \tag{5.11}$$

and prove the next lemma about the function.

Lemma 5.12 *For* $1<\vec{a}\vec{x}\leqslant k$,

(a). $\left\{\alpha^{\vec{a}}_{\vec{x}}(i) \;\middle|\; 0\leqslant i\leqslant |P^{\vec{a}}_{\vec{x}-2\vec{a}}|+\vec{a}\vec{x}\right\}=\left\{i \;\middle|\; 1\leqslant i\leqslant |P^{\vec{a}}_{\vec{x}-\vec{a}}|+\vec{a}\vec{x}+1\right\}-\left\{\beta^{\vec{a}}_{\vec{x}}(i) \;\middle|\; 1\leqslant i\leqslant k+1\right\}$;
(b). $\left\{\beta^{\vec{a}}_{\vec{x}}(i) \;\middle|\; \vec{a}\vec{x}\leqslant i\leqslant k+1\right\}\cup\left\{\alpha^{\vec{a}}_{\vec{x}}\left(\beta^{\vec{a}}_{\vec{x}-\vec{a}}(i)\right) \;\middle|\; 1\leqslant i<\vec{a}\vec{x}-1\right\}>S_{\vec{x}-\vec{a}}\left(Q^{\vec{a}}_{\vec{x}}\right)$ *and*
$\left\{\gamma^{\vec{a}}_{\vec{x}}\left(\beta^{\vec{a}}_{\vec{x}}(i)\right) \;\middle|\; \vec{a}\vec{x}\leqslant i\leqslant k+1\right\}\cup\left\{\gamma^{\vec{a}}_{\vec{x}}\left(\alpha^{\vec{a}}_{\vec{x}}\left(\beta^{\vec{a}}_{\vec{x}-\vec{a}}(i)\right)\right) \;\middle|\; 1\leqslant i<\vec{a}\vec{x}-1\right\}>R_{\vec{x}-\vec{a}}\left(Q^{\vec{a}}_{\vec{x}}\right)$;

(c). $\left\{\beta^{\vec{a}}_{\vec{x}}(i) \;\middle|\; 1\leqslant i<\vec{a}\vec{x}\right\} > S_{\vec{x}-\vec{a}}\left(\left\{A^{\vec{a}}_{\vec{x}}(i) \;\middle|\; 1\leqslant i<\vec{a}\vec{x}\right\}\right)$ *and*
$\gamma^{\vec{a}}_{\vec{x}}\left(\left\{\beta^{\vec{a}}_{\vec{x}}(i) \;\middle|\; 1\leqslant i<\vec{a}\vec{x}\right\}\right) > R_{\vec{x}-\vec{a}}\left(\left\{A^{\vec{a}}_{\vec{x}}(i) \;\middle|\; 1\leqslant i<\vec{a}\vec{x}\right\}\right)$,
(d). $\left\{i \;\middle|\; |P^{\vec{a}}_{\vec{x}}|-k+\vec{a}\vec{x}<i\leqslant|P^{\vec{a}}_{\vec{x}}|\right\} > S_{\vec{x}-\vec{a}}\left(\left\{A^{\vec{a}}_{\vec{x}}(i) \;\middle|\; \vec{a}\vec{x}\leqslant i<k\right\}\right)$ *and*
$\gamma^{\vec{a}}_{\vec{x}}\left(\left\{i \;\middle|\; |P^{\vec{a}}_{\vec{x}}|-k+\vec{a}\vec{x}<i\leqslant|P^{\vec{a}}_{\vec{x}}|\right\}\right) > R_{\vec{x}-\vec{a}}\left(\left\{A^{\vec{a}}_{\vec{x}}(i) \;\middle|\; \vec{a}\vec{x}\leqslant i<k\right\}\right)$.

PROOF. Note that for $\vec{a}\vec{x}>1$ we have $A^{\vec{a}}_{\vec{x}} = M^{\vec{a}}_{\vec{x}}$,

$$\left|P^{\vec{a}}_{\vec{x}}\right| \geqslant \left|P^{-\vec{a}}_{\vec{x}-4\vec{a}}\right| > \left|P^{-\vec{a}}_{\vec{x}-2\vec{a}}\right| > \left|P^{-\vec{a}}_{\vec{x}-\vec{a}}\right| \text{ and } \left|P^{\vec{a}}_{\vec{x}-\vec{a}}\right| \geqslant \left|P^{-\vec{a}}_{\vec{x}-3\vec{a}}\right| > \left|P^{-\vec{a}}_{\vec{x}-2\vec{a}}\right|.$$

(a). The assertion follows from the deduction $\left\{\alpha^{\vec{a}}_{\vec{x}}(i) \;\middle|\; 0\leqslant i\leqslant|P^{\vec{a}}_{\vec{x}-2\vec{a}}|+\vec{a}\vec{x}\right\}$
$= \left\{\alpha^{\vec{a}}_{\vec{x}}(i) \;\middle|\; 0\leqslant i\leqslant|P^{-\vec{a}}_{\vec{x}-\vec{a}}|\right\}\cup\left\{\alpha^{\vec{a}}_{\vec{x}}(i) \;\middle|\; |P^{-\vec{a}}_{\vec{x}-\vec{a}}|<i\leqslant|P^{\vec{a}}_{\vec{x}-2\vec{a}}|\right\}\cup\left\{\alpha^{\vec{a}}_{\vec{x}}(i) \;\middle|\; |P^{-\vec{a}}_{\vec{x}-2\vec{a}}|<i\leqslant|P^{\vec{a}}_{\vec{x}-\vec{a}}|+\vec{a}\vec{x}\right\}$
$= \left\{i \;\middle|\; 1\leqslant i\leqslant|P^{-\vec{a}}_{\vec{x}-\vec{a}}|+1\right\} \cup \left\{\left|P^{-\vec{a}}_{\vec{x}-\vec{a}}\right|+2i \;\middle|\; 1\leqslant i\leqslant k+1\right\} \cup \left\{i \;\middle|\; |P^{-\vec{a}}_{\vec{x}-\vec{a}}|+2k+3<i\leqslant|P^{\vec{a}}_{\vec{x}}|+\vec{a}\vec{x}+1\right\}$
$= \left\{i \;\middle|\; 1\leqslant i\leqslant|P^{\vec{a}}_{\vec{x}}|+\vec{a}\vec{x}+1\right\} - \left\{\left|P^{-\vec{a}}_{\vec{x}-\vec{a}}\right|+2i+1 \;\middle|\; 1\leqslant i\leqslant k+1\right\}$.
(b). Taking note of the following five points:
(1) the second terms in both inequalities of **(b)** are empty if $\vec{a}\vec{x}=2$,
(2) $\beta^{\vec{a}}_{\vec{x}}(i+1) = \beta^{\vec{a}}_{\vec{x}}(i)+2$ for $\vec{a}\vec{x}>1$,
(3) $\alpha^{\vec{a}}_{\vec{x}}\left(\beta^{\vec{a}}_{\vec{x}-\vec{a}}(1)\right) = \beta^{\vec{a}}_{\vec{x}-\vec{a}}(1)+k+2 = \beta^{\vec{a}}_{\vec{x}}(k+1)+3$ for $\vec{a}\vec{x}>2$,
(4) $\alpha^{\vec{a}}_{\vec{x}}\left(\beta^{\vec{a}}_{\vec{x}-\vec{a}}(i+1)\right) = \beta^{\vec{a}}_{\vec{x}-\vec{a}}(i+1)+k+2 = \alpha^{\vec{a}}_{\vec{x}}\left(\beta^{\vec{a}}_{\vec{x}-\vec{a}}(i)\right)+2$, and
(5) Lemma 5.8.**b** and Lemma 5.9.**b**,
to prove the two inequalities, we need only to justify that
$\beta^{\vec{a}}_{\vec{x}}(\vec{a}\vec{x}) > 3k-1$ *and* $\gamma^{\vec{a}}_{\vec{x}}\left(\beta^{\vec{a}}_{\vec{x}}(\vec{a}\vec{x})\right) > 4k$ *for* $1<\vec{a}\vec{x}\leqslant k$. It is true because
$\beta^{\vec{a}}_{\vec{x}}(\vec{a}\vec{x}) = \left|P^{-\vec{a}}_{\vec{x}-\vec{a}}\right|+1+2\vec{a}\vec{x} = (k-\vec{a}\vec{x}+1)(k+1)+1+2\vec{a}\vec{x}$
$= k^2+2k+2-\vec{a}\vec{x}(k-1) \geqslant k^2+2k+2-k(k-1) = 3k+2$ and
$\gamma^{\vec{a}}_{\vec{x}}\left(\beta^{\vec{a}}_{\vec{x}}(\vec{a}\vec{x})\right) = \beta^{\vec{a}}_{\vec{x}}(\vec{a}\vec{x}) + \left|P^{-\vec{a}}_{\vec{x}-\vec{a}}\right| \geqslant 3k+2+k+1 = 4k+3$.
(c). Note that for $\vec{a}\vec{x}>0$ we have $A^{\vec{a}}_{\vec{x}} = M^{\vec{a}}_{\vec{x}}$, implying

$$S_{\vec{x}-\vec{a}}\left(A^{\vec{a}}_{\vec{x}}(i)\right) \leqslant k+2i \text{ and } R_{\vec{x}-\vec{a}}\left(A^{\vec{a}}_{\vec{x}}(i)\right) \leqslant 2k+2i+1$$

by Lemmas 5.8 and 5.9. The assertions in this case follow from
$\beta^{\vec{a}}_{\vec{x}}(i) = \left|P^{-\vec{a}}_{\vec{x}-\vec{a}}\right|+1+2i \geqslant k+2+2i > k+2i \geqslant S_{\vec{x}-\vec{a}}\left(A^{\vec{a}}_{\vec{x}}(i)\right)$ and
$\gamma^{\vec{a}}_{\vec{x}}\left(\beta^{\vec{a}}_{\vec{x}}(i)\right) = \beta^{\vec{a}}_{\vec{x}}(i) + \left|P^{-\vec{a}}_{\vec{x}-\vec{a}}\right| \geqslant 2k+2i+1 \geqslant R_{\vec{x}-\vec{a}}\left(A^{\vec{a}}_{\vec{x}}(i)\right)$.
(d). The assertions follow from that for $1\leqslant i<k$ and $2\leqslant\vec{a}\vec{x}\leqslant k$ we have
$\left|P^{\vec{a}}_{\vec{x}}\right| - k + \vec{a}\vec{x} + i \geqslant \left|P^{-\vec{a}}_{\vec{x}-4\vec{a}}\right| - k + \vec{a}\vec{x} + i = \left|P^{-\vec{a}}_{\vec{x}-\vec{a}}\right| + 2k + 3 + \vec{a}\vec{x} + i$
$> k+2(\vec{a}\vec{x}-1+i) \geqslant S_{\vec{x}-\vec{a}}\left(A^{\vec{a}}_{\vec{x}}(\vec{a}\vec{x}-1+i)\right)$ and
$\gamma^{\vec{a}}_{\vec{x}}\left(\left|P^{\vec{a}}_{\vec{x}}\right| - k + \vec{a}\vec{x} + i\right) = \gamma^{\vec{a}}_{\vec{x}}\left(\left|P^{\vec{a}}_{\vec{x}-\vec{a}}\right| + \vec{a}\vec{x} + i + 1\right)$
$= \left|P^{\vec{a}}_{\vec{x}-\vec{a}}\right| + \vec{a}\vec{x} + i + 1 + \left|P^{-\vec{a}}_{\vec{x}-\vec{a}}\right| = 2k(k+1) + \vec{a}\vec{x} + i + 1$
$> 2k + 2(\vec{a}\vec{x}-1+i) + 1 \geqslant R_{\vec{x}-\vec{a}}\left(A^{\vec{a}}_{\vec{x}}(\vec{a}\vec{x}-1+i)\right)$. □

The next theorem is now ready for the proof, (b) of which implies Proposition 5.1 for the case $\vec{a}\vec{x} > 1$. Thus the proof will complete the whole proof for Theorem 3.1 and close this section.

Theorem 5.13 *For* $0 < \vec{a}\vec{x} \leqslant k$,
(a). $S_{\vec{x}}\left(P^{\vec{a}}_{\vec{x}} - \left\{A^{\vec{a}}_{\vec{x}}(i) \;\middle|\; \vec{a}\vec{x} \leqslant i < k\right\}\right) = \left\{i \;\middle|\; 1 \leqslant i \leqslant |P^{\vec{a}}_{\vec{x}}| - k + \vec{a}\vec{x}\right\}$,
(b). $S_{\vec{x}}\left(P^{\vec{a}}_{\vec{x}}\right) = \left\{i \;\middle|\; 1 \leqslant i \leqslant |P^{\vec{a}}_{\vec{x}}|\right\}$,
(c). $S_{\vec{x}}\left(\left\{A^{\vec{a}}_{\vec{x}}(i) \;\middle|\; \vec{a}\vec{x} \leqslant i < k\right\}\right) = \left\{i \;\middle|\; |P^{\vec{a}}_{\vec{x}}| - k + \vec{a}\vec{x} < i \leqslant |P^{\vec{a}}_{\vec{x}}|\right\}$,
(d). $S_{\vec{x}}\left(\left\{A^{\vec{a}}_{\vec{x}}(i) \;\middle|\; 1 \leqslant i < \vec{a}\vec{x}\right\}\right) \geqslant \left\{\beta^{\vec{a}}_{\vec{x}}(i) \;\middle|\; 1 \leqslant i < \vec{a}\vec{x}\right\}$.

PROOF. By Theorem 5.11, all of the four assertions are true for $\vec{a}\vec{x} = 1$. Suppose, for induction on $\vec{a}\vec{x}$, they are true for $\vec{a}(\vec{x} - \vec{a}) = \vec{a}\vec{x} - 1$. Then for $1 < \vec{a}\vec{x} \leqslant k$ we have $\left|A^{\vec{a}}_{\vec{x}}\right| = \left|M^{\vec{a}}_{\vec{x}}\right| = \left|G^{\vec{a}}_{\vec{x}+\vec{a}}\right| = \left|G^{\vec{a}}_{\vec{x}-\vec{a}}\right| = k - 1$, $A^{\vec{a}}_{\vec{x}} = M^{\vec{a}}_{\vec{x}}$, $G^{\vec{a}}_{\vec{x}} = A^{\vec{a}}_{\vec{x}-\vec{a}}$, $A^{\vec{a}}_{\vec{x}} \cap A^{\vec{a}}_{\vec{x}-\vec{a}} = \phi$, $A^{\vec{a}}_{\vec{x}} \cap \left(P^{\vec{0}}_{\vec{x}-\vec{a}} \cup P^{\vec{a}}_{\vec{x}-\vec{a}}\right) = \phi$, and
$P^{\vec{a}}_{\vec{x}} - \left\{A^{\vec{a}}_{\vec{x}}(i) \;\middle|\; \vec{a}\vec{x} \leqslant i < k\right\} = \left(P^{\vec{0}}_{\vec{x}-\vec{a}} \cup P^{\vec{a}}_{\vec{x}-\vec{a}} - A^{\vec{a}}_{\vec{x}-\vec{a}}\right) \cup Q^{\vec{a}}_{\vec{x}} \cup \left\{A^{\vec{a}}_{\vec{x}}(i) \;\middle|\; 1 \leqslant i < \vec{a}\vec{x}\right\}$.
So, $S_{\vec{x}-\vec{a}}\left(P^{\vec{0}}_{\vec{x}-\vec{a}} \cup P^{\vec{a}}_{\vec{x}-\vec{a}} - A^{\vec{a}}_{\vec{x}-\vec{a}}\right)$
$= S_{\vec{x}-\vec{a}}\left(P^{\vec{0}}_{\vec{x}-\vec{a}} \cup \left(P^{\vec{a}}_{\vec{x}-\vec{a}} - \left\{A^{\vec{a}}_{\vec{x}-\vec{a}}(i) \;\middle|\; \vec{a}(\vec{x}-\vec{a}) \leqslant i < k\right\}\right) - \left\{A^{\vec{a}}_{\vec{x}-\vec{a}}(i) \;\middle|\; 1 \leqslant i < \vec{a}(\vec{x}-\vec{a})\right\}\right)$
$= \left\{i \;\middle|\; 0 \leqslant i \leqslant |P^{\vec{a}}_{\vec{x}-\vec{a}}| - k + \vec{a}(\vec{x}-\vec{a})\right\} - S_{\vec{x}-\vec{a}}\left(\left\{A^{\vec{a}}_{\vec{x}-\vec{a}}(i) \;\middle|\; 1 \leqslant i < \vec{a}(\vec{x}-\vec{a})\right\}\right)$. The last equality is by the induction assumption for **(a)**.

As $\vec{a}\vec{x} > 1$ and $P^{\vec{0}}_{\vec{x}-\vec{a}} \cup P^{\vec{a}}_{\vec{x}-\vec{a}} - A_{\vec{x}-\vec{a}} \subseteq P^{\vec{0}}_{\vec{x}-\vec{a}} \cup P^{\vec{a}}_{\vec{x}-\vec{a}}$ is a prior subset of $P^{\vec{a}}_{\vec{x}}$, by Corollary 5.4

$$\begin{aligned}
&S_{\vec{x}}\left(P^{\vec{0}}_{\vec{x}-\vec{a}} \cup P^{\vec{a}}_{\vec{x}-\vec{a}} - A^{\vec{a}}_{\vec{x}-\vec{a}}\right) = \alpha^{\vec{a}}_{\vec{x}}\left(S_{\vec{x}-\vec{a}}\left(P^{\vec{0}}_{\vec{x}-\vec{a}} \cup P^{\vec{a}}_{\vec{x}-\vec{a}} - A^{\vec{a}}_{\vec{x}-\vec{a}}\right)\right) \\
&= \alpha^{\vec{a}}_{\vec{x}}\left(\left\{i \;\middle|\; 0 \leqslant i \leqslant |P^{\vec{a}}_{\vec{x}-\vec{a}}| - k + \vec{a}(\vec{x}-\vec{a})\right\}\right) - \alpha^{\vec{a}}_{\vec{x}}\left(\left\{S_{\vec{x}-\vec{a}}\left(A^{\vec{a}}_{\vec{x}-\vec{a}}(i)\right) \;\middle|\; 1 \leqslant i < \vec{a}(\vec{x}-\vec{a})\right\}\right). \\
&= \left\{\alpha^{\vec{a}}_{\vec{x}}(i) \;\middle|\; 0 \leqslant i \leqslant |P^{\vec{a}}_{\vec{x}-2\vec{a}}| + \vec{a}\vec{x}\right\} - \left\{\alpha^{\vec{a}}_{\vec{x}}\left(S_{\vec{x}-\vec{a}}\left(A^{\vec{a}}_{\vec{x}-\vec{a}}(i)\right)\right) \;\middle|\; 1 \leqslant i < \vec{a}(\vec{x}-\vec{a})\right\}.
\end{aligned}$$

Applying Lemma 5.12(a) to this, we get

$$\begin{aligned}
S_{\vec{x}}\left(P^{\vec{0}}_{\vec{x}-\vec{a}} \cup P^{\vec{a}}_{\vec{x}-\vec{a}} - A^{\vec{a}}_{\vec{x}-\vec{a}}\right) &= \left\{i \;\middle|\; 1 \leqslant i \leqslant |P^{\vec{a}}_{\vec{x}}| - k + \vec{a}\vec{x}\right\} - \\
&\left(\left\{\beta^{\vec{a}}_{\vec{x}}(i) \;\middle|\; 1 \leqslant i \leqslant k+1\right\} \cup \left\{\alpha^{\vec{a}}_{\vec{x}}\left(S_{\vec{x}-\vec{a}}\left(A^{\vec{a}}_{\vec{x}-\vec{a}}(i)\right)\right) \;\middle|\; 1 \leqslant i < \vec{a}(\vec{x}-\vec{a})\right\}\right). \qquad (5.12)
\end{aligned}$$

Let $J = \left\{\beta^{\vec{a}}_{\vec{x}}(i) \;\middle|\; 1 \leqslant i \leqslant k+1\right\} \cup \left\{\alpha^{\vec{a}}_{\vec{x}}\left(S_{\vec{x}-\vec{a}}\left(A^{\vec{a}}_{\vec{x}-\vec{a}}(i)\right)\right) \;\middle|\; 1 \leqslant i < \vec{a}(\vec{x}-\vec{a})\right\}$, then (5.12) guarantees that $J \cap S_{\vec{x}}\left(P^{\vec{0}}_{\vec{x}-\vec{a}} \cup P^{\vec{a}}_{\vec{x}-\vec{a}} - A^{\vec{a}}_{\vec{x}-\vec{a}}\right) = \phi$. Thus, combining Lemma 5.12(b), Lemma 5.12(c) and the induction assumption for **(d)**, it can be seen that Observation 1 is applicable to J and the two prior subsets $F_1 = P^{\vec{0}}_{\vec{x}-\vec{a}} \cup P^{\vec{a}}_{\vec{x}-\vec{a}} - A^{\vec{a}}_{\vec{x}-\vec{a}}$ and $F_2 = \left(P^{\vec{0}}_{\vec{x}-\vec{a}} \cup P^{\vec{a}}_{\vec{x}-\vec{a}} - A^{\vec{a}}_{\vec{x}-\vec{a}}\right) \cup Q^{\vec{a}}_{\vec{x}} \cup \left\{A^{\vec{a}}_{\vec{x}}(i) \;\middle|\; 1 \leqslant i < \vec{a}\vec{x}\right\}$, giving us

$$S_{\vec{x}}\left(Q^{\vec{a}}_{\vec{x}} \cup \left\{A^{\vec{a}}_{\vec{x}}(i) \;\middle|\; {\scriptstyle 1\leqslant i<\vec{a}\vec{x}}\right\}\right)$$
$$\leqslant \left\{\beta^{\vec{a}}_{\vec{x}}(i) \;\middle|\; {\scriptstyle 1\leqslant i\leqslant k+1}\right\} \cup \left\{\alpha^{\vec{a}}_{\vec{x}}\left(S_{\vec{x}-\vec{a}}\left(A^{\vec{a}}_{\vec{x}-\vec{a}}(i)\right)\right) \;\middle|\; {\scriptstyle 1\leqslant i<\vec{a}(\vec{x}-\vec{a})}\right\}. \tag{5.13}$$

In the above, the equality holds because (5.12) implies that

$$\left(\left\{\beta^{\vec{a}}_{\vec{x}}(i) \;\middle|\; {\scriptstyle 1\leqslant i\leqslant k+1}\right\} \cup \left\{\alpha^{\vec{a}}_{\vec{x}}\left(S_{\vec{x}-\vec{a}}\left(A^{\vec{a}}_{\vec{x}-\vec{a}}(i)\right)\right) \;\middle|\; {\scriptstyle 1\leqslant i<\vec{a}(\vec{x}-\vec{a})}\right\}\right) \subseteq \left\{i \;\middle|\; {\scriptstyle 1\leqslant i\leqslant \left|P^{\vec{a}}_{\vec{x}}\right|-k+\vec{a}\vec{x}}\right\}$$

and that the number that is smaller than those in the lefthand of the above has already been assigned to some message of $P^{\vec{0}}_{\vec{x}-\vec{a}} \cup P^{\vec{a}}_{\vec{x}-\vec{a}} - A^{\vec{a}}_{\vec{x}-\vec{a}}$. Then **(a)** is a straightforward combination of (5.12) and the equality version of (5.13). **(d)** can also be derived from the equality of (5.13) because for $1 \leqslant i < \vec{a}(\vec{x}-\vec{a})$, by the induction assumption for **(d)**,

$$\alpha^{\vec{a}}_{\vec{x}}\left(S_{\vec{x}-\vec{a}}\left(A^{\vec{a}}_{\vec{x}-\vec{a}}(i)\right)\right) \geqslant \alpha^{\vec{a}}_{\vec{x}}\left(\beta^{\vec{a}}_{\vec{x}-\vec{a}}(i)\right) > \beta^{\vec{a}}_{\vec{x}}(k+1).$$

From **(a)**, now assuming J to be $\left\{i \;\middle|\; {\scriptstyle \left|P^{\vec{a}}_{\vec{x}}\right|-k+\vec{a}\vec{x}<i\leqslant\left|P^{\vec{a}}_{\vec{x}}\right|}\right\}$, and by Lemma 5.12(d), it can be seen that Observation 1 is applicable to the two prior subsets $F_1 = P^{\vec{a}}_{\vec{x}} - \left\{A^{\vec{a}}_{\vec{x}}(i) \;\middle|\; {\scriptstyle \vec{a}\vec{x}\leqslant i<k}\right\}$ and $F_2 = P^{\vec{a}}_{\vec{x}}$, giving us **(c)**. Finally, combining **(c)** with **(a)**, **(b)** follows. □

6 The Proof of Theorem 4.1

The proof for Theorem 4.1 is analogous to the one for Theorem 3.1. In Section 5, the arguments preceding Section 5.1 (from the beginning to Observation 1) are readily applicable to the even n case. The other arguments can be adapted to even n, and the adaption is simple.

Seeing that

$$\left|P^{\vec{a}}_{\vec{x}}\right| \leqslant \left|P^{-\vec{a}}_{\vec{x}-\vec{a}}\right| \overset{n\ odd}{\Longleftrightarrow} \vec{a}\vec{x} \leqslant 0$$

while

$$\left|P^{\vec{a}}_{\vec{x}}\right| \leqslant \left|P^{-\vec{a}}_{\vec{x}-\vec{a}}\right| \overset{n\ even}{\Longleftrightarrow} \left(\vec{a}\vec{x} \leqslant 1 \bigwedge \Delta\vec{a}\vec{x} \leqslant 0\right) \bigvee \left(\vec{a}\vec{x} < 1 \bigwedge \Delta\vec{a}\vec{x} > 0\right)$$

we know that "$(\vec{a}\vec{x} \leqslant 1 \bigwedge \Delta\vec{a}\vec{x} \leqslant 0) \bigvee (\vec{a}\vec{x} < 1 \bigwedge \Delta\vec{a}\vec{x} > 0)$" should correspond to "$\vec{a}\vec{x} \leqslant 0$". Substituting either one in a claim with its correspondence is likely to switch the claim between odd n and even n cases; and moreover, simply substituting them with "$\left|P^{\vec{a}}_{\vec{x}}\right| \leqslant \left|P^{-\vec{a}}_{\vec{x}-\vec{a}}\right|$" would generalize the claim for both odd n and even n. Some other similar correspondences are:
$\vec{a}\vec{x} > 0 \overset{odd\ n}{\Longleftrightarrow} \left|P^{\vec{a}}_{\vec{x}}\right| > \left|P^{-\vec{a}}_{\vec{x}-\vec{a}}\right| \overset{even\ n}{\Longleftrightarrow} \vec{a}\vec{x} > 1 \bigwedge \Delta\vec{a}\vec{x} \leqslant 0 \bigvee \vec{a}\vec{x} > 0 \bigwedge \Delta\vec{a}\vec{x} > 0$,
$\vec{a}\vec{x} > 1 \overset{odd\ n}{\Longleftrightarrow} \left|P^{\vec{a}}_{\vec{x}-\vec{a}}\right| > \left|P^{-\vec{a}}_{\vec{x}-2\vec{a}}\right| \overset{even\ n}{\Longleftrightarrow} \vec{a}\vec{x} > 2 \bigwedge \Delta\vec{a}\vec{x} \leqslant 0 \bigvee \vec{a}\vec{x} > 1 \bigwedge \Delta\vec{a}\vec{x} > 0$,

$\vec{a}\vec{x} = 1 \overset{odd\ n}{\Longleftrightarrow} \left|P^{\vec{a}}_{\vec{x}}\right| > \left|P^{-\vec{a}}_{\vec{x}-\vec{a}}\right| \bigwedge \left|P^{\vec{a}}_{\vec{x}-\vec{a}}\right| \leqslant \left|P^{-\vec{a}}_{\vec{x}-2\vec{a}}\right| \overset{even\ n}{\Longleftrightarrow} \vec{a}\vec{x} = 2 \bigwedge \Delta\vec{a}\vec{x} \leqslant 0 \bigvee \vec{a}\vec{x} = 1 \bigwedge \Delta\vec{a}\vec{x} > 0$.

All the lemmas, corollaries, and theorems in Section 5 can be adapted to even n, or generalized for both odd n and even n. These adapted or generalized versions can be proved in the same ways. Function $\alpha^{\vec{a}}_{\vec{x}}$ can be generalized as

$$\alpha^{\vec{a}}_{\vec{x}}(s) = \begin{cases} s+1 & \left|P^{\vec{a}}_{\vec{x}}\right| \leqslant \left|P^{-\vec{a}}_{\vec{x}-\vec{a}}\right| \text{ or } s \leqslant \left|P^{-\vec{a}}_{\vec{x}-\vec{a}}\right|, \\ 2s - \left|P^{-\vec{a}}_{\vec{x}-\vec{a}}\right| & \left|P^{\vec{a}}_{\vec{x}}\right| > \left|P^{-\vec{a}}_{\vec{x}-\vec{a}}\right|,\ \left|P^{-\vec{a}}_{\vec{x}-\vec{a}}\right| < s \leqslant \left|P^{-\vec{a}}_{\vec{x}-2\vec{a}}\right|, \\ s + \left|P^{-\vec{a}}_{\vec{x}-2\vec{a}}\right| - \left|P^{-\vec{a}}_{\vec{x}-\vec{a}}\right| + 1 & \left|P^{\vec{a}}_{\vec{x}}\right| > \left|P^{-\vec{a}}_{\vec{x}-\vec{a}}\right|,\ s > \left|P^{-\vec{a}}_{\vec{x}-2\vec{a}}\right|. \end{cases}$$

Function $\beta^{\vec{a}}_{\vec{x}}$ can be adapted to even n case as

$$\beta^{\vec{a}}_{\vec{x}}(i) = \begin{cases} \left|P^{\vec{a}}_{\vec{x}-\vec{a}}\right| + 1 & \vec{a}\vec{x}=2,\ \Delta\vec{a}\vec{x}\leqslant 0,\ i=1, \\ \left|P^{\vec{a}}_{\vec{x}-\vec{a}}\right| + 1 + i & \left|P^{\vec{a}}_{\vec{x}-\vec{a}}\right| \leqslant \left|P^{-\vec{a}}_{\vec{x}-\vec{a}}\right|, \\ \left|P^{-\vec{a}}_{\vec{x}-\vec{a}}\right| + 1 + 2i & \left|P^{\vec{a}}_{\vec{x}-\vec{a}}\right| > \left|P^{-\vec{a}}_{\vec{x}-\vec{a}}\right|. \end{cases}$$

We only mention the revised versions of the basic claims. (5.2) in Lemma 5.6 can be generalized to

$$S_{\vec{x}}(\vec{x} - i\vec{a}) = \begin{cases} i & i-1 \leqslant \left|P^{-\vec{a}}_{\vec{x}-\vec{a}}\right|, \\ 2(i-1) - \left|P^{-\vec{a}}_{\vec{x}-\vec{a}}\right| & \text{otherwise} \end{cases}$$

or adapted to even n case as

$$S_{\vec{x}}(\vec{x} - i\vec{a}) = \begin{cases} 2i-k-3 & \Delta\vec{a}\vec{x}\leqslant 0,\ \vec{a}\vec{x}=k, \\ 2i-k-2 & \Delta\vec{a}\vec{x}>0,\ \vec{a}\vec{x}=k, \\ i & otherwise. \end{cases}$$

The former one is general, but the latter one is convenient for use. Lemma 5.2 and Corollaries 5.3 and 5.4 need only to be generalized by substituting $\vec{a}\vec{x} \leqslant 0$, $\vec{a}\vec{x} > 1$, and $\vec{a}\vec{x} \neq 1$ respectively with $\left|P^{\vec{a}}_{\vec{x}}\right| \leqslant \left|P^{-\vec{a}}_{\vec{x}-\vec{a}}\right|$, $\left|P^{\vec{a}}_{\vec{x}-\vec{a}}\right| > \left|P^{-\vec{a}}_{\vec{x}-2\vec{a}}\right|$, and $\left|P^{\vec{a}}_{\vec{x}}\right| \leqslant \left|P^{-\vec{a}}_{\vec{x}-\vec{a}}\right| \bigvee \left|P^{\vec{a}}_{\vec{x}-\vec{a}}\right| > \left|P^{-\vec{a}}_{\vec{x}-2\vec{a}}\right|$.

Lemma 5.5 remains intact, but the claims of Lemma 5.7 change to
(a). $S_{\vec{x}}\left(I^{\vec{a}}_{\vec{x}-\vec{a}\vec{x}\vec{a}}(k-1)\right) \leqslant \left\{k+2i-2 \;\middle|\; 1\leqslant i\leqslant k-1\right\}$ for $\vec{a}\vec{x} \geqslant 0$,
(b). $S_{\vec{x}}\left(\Gamma^{\vec{a}}_{\vec{x}}\right) \leqslant \left\{3k+2i-4 \;\middle|\; 1\leqslant i\leqslant |\Gamma^{\vec{a}}_{\vec{x}}|\right\}$.

Thus, the assertions in Lemmas 5.8 are changed accordingly to
(a). $S_{\vec{x}-\vec{a}}\left(Q^{\vec{a}}_{\vec{x}}\right) \leqslant \left\{i \;\middle|\; 1\leqslant i\leqslant |Q^{\vec{a}}_{\vec{x}}|\right\}$ for $-k \leqslant \vec{a}\vec{x} \leqslant 0$,

(b). $S_{\vec{x}-\vec{a}}\left(Q_{\vec{x}}^{\vec{a}}\right) \leqslant \left\{3k+2i-4 \;\middle|\; 1\leqslant i\leqslant k\right\}$ for $0<\vec{a}\vec{x}\leqslant k$,
(c). $S_{\vec{x}-\vec{a}}\left(M_{\vec{x}}^{\vec{a}}\right) \leqslant \left\{k+2i-2 \;\middle|\; 1\leqslant i\leqslant |M_{\vec{x}}^{\vec{a}}|\right\}$;

and those of Lemma 5.9 are changed similarly to
(a). $R_{\vec{x}-\vec{a}}\left(Q_{\vec{x}}^{\vec{a}}\right) \leqslant \left\{2i \;\middle|\; 1\leqslant i\leqslant |Q_{\vec{x}}^{\vec{a}}|\right\}$ for $-k\leqslant \vec{a}\vec{x}\leqslant 0$,
(b). $R_{\vec{x}-\vec{a}}\left(Q_{\vec{x}}^{\vec{a}}\right) \leqslant \left\{4k+2i-3 \;\middle|\; 1\leqslant i\leqslant |Q_{\vec{x}}^{\vec{a}}|\right\}$ for $0<\vec{a}\vec{x}\leqslant k$,
(c). $R_{\vec{x}-\vec{a}}\left(M_{\vec{x}}^{\vec{a}}\right) \leqslant \left\{2k+2i-1 \;\middle|\; 1\leqslant i\leqslant |M_{\vec{x}}^{\vec{a}}|\right\}$.

Based on these changes, Theorems 5.10, 5.11, 5.13, as well as Lemma 5.12, can be revised accordingly. As in the odd n case, the three revised theorems lead us to Proposition 5.1 for the even n case.

7 Discussion

We have designed optimal gossiping algorithms for square meshes under the F* and the H* model. The two algorithms are closely related in that over the same edge, they send the same messages in the same order. It seems possible that the ideas presented here can also be applied to higher dimensional square meshes and tori to obtain optimal or near-optimal results. For non-square 2D meshes, by relaxing the rotational symmetry from 90° to 180° (i.e., $R_{\vec{x}}(\vec{m}) = R_{-\vec{x}}(-\vec{m})$ and $\vec{m}\in P_{\vec{x}}^{\vec{a}} \Longleftrightarrow -\vec{m}\in P_{-\vec{x}}^{-\vec{a}}$), the method can be extended to result in fast algorithms.

In Step 3 of Procedure RS, the productive set $W_{\vec{x}}^{\vec{a}}$ and the ordinary set $P_{\vec{x}}^{\vec{a}} - \left(W_{\vec{x}}^{\vec{a}}\cup A_{\vec{x}}^{\vec{a}}\right)$ are subdivided for an easier proof of Theorems 3.1 and 4.1. In fact these subdivisions are not necessary, Step 3 can be

3. For $C = W_{\vec{x}}^{\vec{a}}$, $P_{\vec{x}}^{\vec{a}} - \left(W_{\vec{x}}^{\vec{a}}\cup A_{\vec{x}}^{\vec{a}}\right)$, $A_{\vec{x}}^{\vec{a}}$, respectively, while $C\neq\phi$ repeat ...

For the F* model only, the optimal solution can be simplified (see [27]). The message gathering can be simplified to

$$P_{\vec{x}}^{\vec{a}} = \begin{cases} \phi & \vec{a}\vec{x}=-k, \\ \{\vec{x}\} & \vec{a}=\vec{0}, \\ P_{\vec{x}-\vec{a}}^{\vec{0}} \cup P_{\vec{x}-\vec{a}}^{\vec{a}} \cup Q_{\vec{x}}^{\vec{a}} & otherwise; \end{cases}$$

where $Q_{\vec{x}}^{\vec{a}}$ is the same as (3.4) or (4.1), depending on the parity of n. The F* round assignment for $P_{\vec{x}}^{\vec{a}}$ can be simply determined as

for the edge $\{\vec{x},\ \vec{x}-\vec{a}\}$, let node $\vec{x}$ first collect $P_{\vec{x}-\vec{a}}^{\vec{0}}\cup P_{\vec{x}-\vec{a}}^{\vec{a}}$, then $Q_{\vec{x}}^{\vec{a}}$; within each set, adopt the principle *"first to $\vec{x}-\vec{a}$, first to $\vec{x}$"*, breaking ties arbi-

trarily.

The above idea assures the precedence constraint for the F* GS, which can be proved easily (see [27]).

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References

[1] A. Bagchi, E.F. Schmeichel, and S.L. Hakimi, "Sequential information dissemination by packets", *Networks*, vol. 22, pp. 317–333, 1992.

[2] A. Bagchi, E.F. Schmeichel, and S.L. Hakimi, "Parallel information dissemination by packets", *SIAM J. Computing*, vol. 23, pp. 355–372, 1994.

[3] J.-C. Bermond, T. Kodate, and S. Perennes, "Gossiping in Cayley graphs by packets", *Proc. French-Japanese Conf. on Combinatorics and Computer Science*, LNCS 1120, Springer-Verlag, pp. 301–315, 1995.

[4] J.-C. Bermond, L. Gargano, and S. Perennes, "Optimal sequential gossiping by short messages", *Discrete Applied Mathematics*, vol. 86, pp. 145–155, 1998.

[5] J.-C. Bermond, L. Gargano, A.A. Rescigno, and U. Vaccaro, "Fast gossiping by short messages", *SIAM J. Computing*, vol. 27, pp. 917–941, 1998.

[6] W.J. Dally, and P. Song, "Design of self-timed VLSI multicomputer communication controller", *Proc. Int'l Conf. Computer Design*, pp. 230–234, 1987.

[7] W.J. Dally, "The J-Machine: System support for actors", in *Actors: Knowledge-Based Concurrent Computing*, Hewitt and Agha, eds., 1989.

[8] J. Duato, S. Yalamanchili, and L. Ni, *Interconnection networks: An engineering approach*, Los Alamitos, Calif.: IEEE CS Press, 1997.

[9] P. Fraigniaud and E. Lizard, "Methods and problems of communication in usual networks", *Discrete Applied Math.*, vol. 53, pp. 79–134, 1994.

[10] S. Fujita and M. Yamashita, "Fast gossiping on square mesh computers", *Information Processing Letters*, vol. 48, pp. 127–130, 1993.

[11] S.M. Hedetniemi, S.T. Hedetniemi, and A. Liestman, "A survey of gossiping and broadcasting in communication networks", *Networks*, vol. 18, pp. 319–349, 1988.

[12] J. Hromkovič, R. Klasing, B. Monien, and R. Peine, "Dissemination of information in interconnection networks (Broadcasting & Gossiping)", in *Combinatorial Network Theory*, pp. 125–212, D.-Z. Du and D.F. Hsu, eds., 1996.

[13] J. Hromkovič, R. Klasing, A. Pelc, P. Ružička, and W. Unger, *Dissemination of information in communication networks: Broadcasting, gossiping, leader election, and fault-tolerance*, Springer, 2005.

[14] B.H.H. Juurlink, J.F. Sibeyn, and P.S. Rao, "Gossiping on meshes and tori", *IEEE Trans. parallel and distributed systems*, vol. 9, no. 6, pp. 513–525, 1998.

[15] M. Kaufmann and J.F. Sibeyn, "Randomized multipacket routing and sorting on meshes", *Algorithmca*, vol. 17, pp. 224–244, 1997.

[16] D.W. Krumme, G. Cybenko, and K.N. Venkataraman, "Gossiping in minimum time", *SIAM J. Computing*, vol. 21, no.1, pp. 111–139, 1992.

[17] D.W. Krumme, "Fast gossiping for the hypercube", *SIAM J. Computing*, vol. 21, no.2, pp. 365–380, 1992.

[18] F.C.M. Lau and S.H. Zhang, "Fast gossiping in square meshes/tori with bounded-size packets", *IEEE Trans. on Parallel and Distributed Systems*, vol. 13, no. 4, pp. 349–358, 2002.

[19] F.C.M. Lau and S.H. Zhang, "Optimal gossiping in paths and cycles", *J. of Discrete Algorithms*, vol. 1, no. 5-6, pp. 461–475, 2003.

[20] S.L. Lillevik, "The touchstone 30-Gigaflop DELTA prototype", *Proc. Sixth Distributed Memory Computing Conf.*, pp. 671–677, 1991.

[21] M.D. May, P.W. Thompson, and P.H. Welch, *Networks, routers, and transputers*, Amsterdam: IOS Press, 1993.

[22] P.S. Rao and G. Mouney, "Data communications in parallel block predictor-corrector methods for solving ODEs", Tech. Rep. LAAS-CNRS, France, 1995.

[23] M. Soch and P. Tvrdik, "Optimal gossiping in store-and-forward noncombining 2-D tori", *Proc. Euro-Par '97*, LNCS 1300, Springer-Verlag, pp. 234–241, 1997.

[24] M. Soch and P. Tvrdik, "Optimal gossiping in noncombining 2D meshes", *Proc. Int'l Colloq. Structural Information and Comm. Complexity (SIROCCO '97)*, pp. 253–265, 1997.

[25] M. Soch and P. Tvrdik, "Optimal gossiping noncombining 2D meshes and tori", Tech. Rep. DC-97-04, Dept. of CS&E, Czech Technical Univ., 1997.

[26] C.L. Seitz, "The architecture and programming of the Ametek series 2010 multicomputer", *Proc. Third Conf. Hypercube Concurrent Computers and Applications*, pp. 33–36, 1988.

[27] R. Wang, and F.C.M Lau, "Optimal gossiping in square meshes in all-port mode and with short packets", *6th Internat. Colloq. on Structural Information and Communication Complexity (SIROCCO '05)*, LNCS 3499, Springer-Verlag, pp. 292–306, 2005.