Unbounded One-Way Trading on Distributions with Monotone Hazard Rate

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Abstract. One-way trading is a fundamental problem in the online algorithms. A seller has some product to be sold to a sequence of buyers $\{u_1, u_2, \ldots\}$ in an online fashion and each buyer u_i is associated with his accepted unit price p_i, which is known to the seller on the arrival of u_i. The seller needs to decide the amount of products to be sold to u_i at the then-prevailing price p_i. The objective is to maximize the total revenue of the seller. In this paper, we study the unbounded one-way trading, i.e., the highest unit price among all buyers is unbounded. We also assume that the highest prices of buyers follow some distribution with monotone hazard rate, which is well-adopted in Economics. We investigate two variants, (1) the distribution is on the highest price among all buyers, and (2) a general variant that the prices of buyers is independent and identically distributed. To measure the performance of the algorithms, the expected competitive ratios, $\mathrm{E}[OPT]/\mathrm{E}[ALG]$ and $\mathrm{E}[OPT/ALG]$, are considered and constant-competitive algorithms are given if the distributions satisfy the monotone hazard rate.

1 Introduction

Revenue maximization is an important problem studied by researchers in the fields of economics, mathematics and computer science. This problem has many variations but generally involves the question of how to sell or assign products (goods or services) to various buyers. The assignment of products includes determining both the price and the amount of products sold to each buyer, which is a fundamental problem related to markets and market mechanisms in economics. Accordingly, there are two ways for a seller to maximize revenue: controlling the selling price and controlling the amount sold.

In this paper, we focus on the design of an online strategy to determine how much should be sold at the prevailing market price (which cannot be controlled by the seller) at different times. This problem was first studied by El-Yaniv et al. [12, 13], which was called named *one-way trading*. In the one-way trading problem, a player has some initial asset (e.g., dollar) to be changed to a target asset (e.g., yen). The exchange rate fluctuates over time. To maximize the revenue, the player must decide the amount of the initial asset to be changed when the exchange rate on each day is known. The offline version of this problem is straightforward as the seller can know all the future information: the seller can simply exchange all initial assets to the target asset on the day with the highest exchange rate. However, in the online version where the player has no knowledge of the future, at no point will the player be sure that the prevailing exchange rate is the highest one. The key features of the one-way trading problem are: (1) the player has no control of the exchange rate which fluctuates over time; (2) the player has no knowledge, or incomplete knowledge, of the future; and (3) the player can decide the amount to be changed only upon the arrival of each rate.

The one-way trading problem studied in [13] is the bounded version, i.e., the range of the exchange rate is in $[m, M]$, where m and M are fixed values. Based on the relationship between m and M, El-Yaniv et al. presented an optimal online algorithm by using a threat-based policy, of which the competitive ratio is $\Theta(\log(M/m))$. If the highest possible rate is unbounded, even for a fixed number of transactions, the threat-based policy cannot be implemented since the ratio between any two rates can be arbitrary large. In the bounded one-way trading problem, the remaining amount of the initial asset after the last transaction will be changed to the target asset with the minimum rate m. However, if the highest possible rate is unbounded, in the worst case, the total revenue is dominated by the revenue from high rates and the revenue from the remaining asset using the minimum rate is very tiny and ignoring this part will hardly affect the performance. For the one-way trading with unbounded value, Chin et al. [10] gave a near optimal algorithm with competitive ratio $O(\log r^*(\log^{(2)} r^*) \ldots (\log^{(h-1)} r^*)(\log^{(h)} r^*)^{1+\epsilon})$ if the value of $r^* = p^*/p_1$, the ratio between the highest market price $p^* = \max_i p_i$ and the first price p_1, is large and satisfies $\log^{(h)} r^* > 1$, where $\log^{(i)} x$ denotes the application of the logarithm function i times to x; otherwise, the algorithm has a constant competitive ratio. A lower bound was also proved in [10]. Given any positive integer h and any one-way trading algorithm A, a sequence of buyers σ with $\log^{(h)} r^* > 1$ exist such that the ratio between the optimal revenue and the revenue obtained by A is at least $\Omega(\log r^*(\log^{(2)} r^*) \ldots (\log^{(h-1)} r^*)(\log^{(h)} r^*))$.

In some sense, the one-way trading problem can be regarded as a *time series search problem*, the objective of which is to find the maximum (or the minimum) value among a sequence of values in an online fashion. For the 1-max-search variant, i.e., determining the highest value among the whole sequence in an online fashion, El-Yaniv et al. [13] presented a randomized $O(\log M/m)$-competitive algorithm if the values fluctuate between m and M; when M/m is unknown in advance, a randomized online algorithm with competitive ratio $O(\log(M/m) \cdot$

$\log^{1+\epsilon}(\log(M/m)))$ can be achieved. In [16], Lorenz et al. gave an optimal online algorithm for the k-search problem, in which the player's target is to find the k highest (or lowest) values among all values in a sequence and the values are chosen from $[m, M]$.

In this paper, we assume that items can be sold fractionally, thus, the amount of items can be normalized to be 1. A sequence of buyers come one after one and each buyer i is associated with a price p_i, which is his accepted unit price for buying the items. Only upon the arrival of a buyer i will his accepted price p_i be known to the seller, who will immediately determine the amount of items to be sold to the buyer with unit price p_i. The objective is to maximize the total revenue of the seller. In the unbounded one-way trading, the range of the accepted prices is in $(0, +\infty)$.

In all previous studies, if there is no information about the future prices, no algorithm achieved a competitive ratio better than a logarithm factor. However, given some partial information about the prices, the performance could be improved greatly. In this paper, we assume that the distribution of the highest accepted price is the partial information that is known. Firstly, assume that the distribution is on the highest price among all buyers, i.e., $\max_i p_i$. We then consider a general variant where the sequence of prices of buyers is *independent and identically distributed* (i.i.d.).

To measure the performance of the online algorithm, the competitive ratio is often used, which denotes the ratio between the result form the online algorithm and the optimal offline algorithm. For the online algorithm with distributions, we use the expected competitive ratio for evaluation. There are mainly two kinds of expectation of competitive ratio, i.e., $\frac{\mathrm{E}[OPT]}{\mathrm{E}[ALG]}$ and $\mathrm{E}[\frac{OPT}{ALG}]$. Both of them are considered with respect to different situations and the values of them may vary a lot. For the former measure, since the expected value of the optimal solution is independent of the algorithm solution, the target is to maximize the expected output of the algorithm.

The paper is organized as follows: Section 2 describes the one-way trading with distributions and the measurement of the algorithm; in Section 3, constant competitive algorithms are given if the distribution is on the highest price among all buyers; in Section 4, we prove that the variant with i.i.d. distribution on each buyer can be reduced to the variant in Section 3, and thus constant-competitive algorithms can be obtained too.

2 One-Way Trading with Distribution

In the one-way trading problem, we may regard the first price as a unit value. This assumption is reasonable since in the remaining part of the price sequence, values lower than the first one could be ignored and will not affect the performance. Let f be the density function and F be the accumulated distribution with respect to the highest price among all buyers. We assume that f and F are continuous in $[1, +\infty)$. Given F, the expected revenue received from the optimal algorithm is

$$\mathrm{E}[OPT] = \int_1^{+\infty} x dF(x) = \int_1^{+\infty} x f(x) dx.$$

El-Yaniv et al. showed that for the bounded one-way trading problem, the adversary can choose the worst distribution on the highest selling price and force the online algorithm to achieve the competitive ratio no less than $\Omega(\log M/m)$ (Theorem 7 in [13]), where the highest price $p \in [m, M]$. This result can be extended to the unbounded one-way trading problem.

Fact 1 *There exists the worst distribution F such that no online algorithm can solve the unbounded one-way trading problem with the competitive ratio better than a logarithm factor if the highest price is drawn from F.*

Proof. The distribution on the bounded one-way trading can be also used as the distribution on the unbounded version such that the probabilities on the highest price higher than M and lower than m are both zero. Thus, setting the distribution F to be the worst distribution w.r.t. the bounded one-way trading implies the competitive ratio of any online algorithm cannot be better than a logarithm factor. □

This negative result is unimportant in reality since most frequently used distributions in economics are far from the worst distribution. If the highest price among all buyers is uniformly distributed, Fujiwara et al. [14] considered the selling strategy according to several measures, e.g., $\mathrm{E}[ALG/OPT]$, $\mathrm{E}[OPT/ALG]$, $\mathrm{E}[ALG]/\mathrm{E}[OPT]$, $\mathrm{E}[OPT]/\mathrm{E}[ALG]$. The algorithms for the average case analysis of the bounded one-way trading are based on the threat-based policy. However, such a strategy does not work for the unbounded variant since the lowest price m and highest price M may not be known in advance.

The hazard rate, a.k.a. *the failure rate*, is the probability of observing an outcome within a neighborhood of some value x, conditional on the outcome being no less than x. The concept of the hazard rate is well-adopted in economics. For example, in English auctions, the hazard rate on x denotes the probability of the auction ending at x, conditional on the bidders' prices reach x. In this paper, we consider *the monotone hazard rate*, which is reasonable and also has been considered in theoretical computer science [8, 17]. Formally speaking,

Definition 1. *(Monotone Hazard Rate). A distribution F with density f is said to satisfy the monotone hazard rate (MHR) if $\frac{1-F(x)}{f(x)}$ is monotonically non-increasing for all $x > 0$.*

3 Distribution on the Highest Price among All Buyers

In this part, we consider the variant that the distribution on the highest price among all buyers is known in advance and satisfies the monotone hazard rate.

3.1 Measure of $\frac{E[OPT]}{E[ALG]}$

The following two lemmas from Chawla et al. [8] and Dhangwatnotai et al. [11] respectively can be regarded as the consequences of Myerson's optimal strategy [18]. They also gave the idea to maximize the algorithm's expected revenue.

Lemma 1. [8] *If the distribution F with density f satisfies MHR, then there exists x_0 such that (1) $x_0(1-F(x_0))$ is maximized, (2) for any $x_0 < x_1 < x_2$, $x_0(1-F(x_0)) > x_1(1-F(x_1)) > x_2(1-F(x_2))$ and, (3) for any $x_0 > x_1 > x_2$, $x_0(1-F(x_0)) > x_1(1-F(x_1)) > x_2(1-F(x_2))$.*

From Lemma 1, it is natural to assign all products to any buyer with value no less than x_0. With probability $1-F(x_0)$, all products are assigned with price no less than x_0, which means the expected revenue from the algorithm is at least $x_0 \cdot (1-F(x_0))$.

Lemma 2. [11] $\mathrm{E}[OPT] = O(x_0 \cdot (1-F(x_0)))$

According to the above two lemmas, the algorithm can be simply described as follows.

Algorithm 1 Online Selling for the measure of $\mathrm{E}[OPT]/\mathrm{E}[ALG]$

1: Let $x_0 = \arg\max_x x \cdot (1-F(x))$
2: Sell the whole product to the first buyer with price no less than x_0.

Thus, we have the following conclusion.

Theorem 1. *When considering the measure of $\frac{\mathrm{E}[OPT]}{\mathrm{E}[ALG]}$, the expected competitive ratio of Algorithm 1 is a constant.*

3.2 Measure of $\mathrm{E}[\frac{OPT}{ALG}]$

For the measure of $\mathrm{E}[\frac{OPT}{ALG}]$, the competitive ratio of Algorithm 1 is unbounded since the seller does not assign any product to the buyer with price less than $\arg\max_x x(1-F(x))$ and the ratio in such case is unbounded. Thus, we have to investigate the intrinsic property and find other way to achieve good performance for this measurement.

Lemma 3. *Given a distribution F satisfying MHR, $h(x) = \frac{1-F(x)}{1-F(2x)}$ is monotone non-decreasing.*

Proof.

$$
\begin{aligned}
h'(x) &= \frac{-(1-F(2x))f(x) + 2(1-F(x))f(2x)}{(1-F(2x))^2} \\
&= \frac{2f(2x)(1-F(x)) - f(x)(1-F(2x))}{(1-F(2x))^2} \\
&= \frac{2f(2x)}{1-F(2x)} \cdot \frac{1-F(x)}{1-F(2x)} - \frac{f(x)}{1-F(2x)}
\end{aligned}
$$

Since F satisfies MHR, i.e., $\frac{1-F(x)}{f(x)} \geq \frac{1-F(2x)}{f(2x)}$, we have $h'(x) \geq 0$, which means that $h(x)$ is monotone non-decreasing. □

From Lemma 1, we know that if the distribution of the highest price satisfies MHR, there exists p such that $p \cdot (1 - F(p))$ is maximized. W.l.o.g., assume that $2^k \leq p < 2^{k+1}$. As mentioned before, if the coming price is no less than p, selling the whole item to this buyer is a good idea. But for the remaining case that the highest price is strictly less than p, the assignment is also critical. In our algorithm, the item is partitioned with respect to the range of the price. Upon the arrival of a buyer, if his price is the first in some range, the corresponding amount of item will be assigned to him. The description of the algorithm is shown in Algorithm 2.

Algorithm 2 Online Selling for the measure of $\mathrm{E}[OPT/ALG]$

1: **if** v is the first value no less than p **then**
2: Assign 1/2 product to this buyer.
3: **else**
4: **if** v is the first value within $[2^{-i} \cdot p, 2^{1-i} \cdot p)$ **then**
5: Assign 2^{-i-1} product to this buyer.
6: **end if**
7: **end if**

Theorem 2. *When considering the measure of* $\mathrm{E}[\frac{OPT}{ALG}]$, *the expected competitive ratio of the above algorithm is a constant.*

Proof. For a sequence of buyers, suppose that the highest price among all buyers is x. The maximal revenue for this sequence is x by assigning the whole product to the buyer with the highest price. Let $ALG(x)$ be the revenue received by the online algorithm on a buyer sequence with the highest price x.

According to the online algorithm, if $x \geq p$, the algorithm assigns 1/2 of a product to a buyer with price no less than p; if $x \in [2^{-i} \cdot p, 2^{1-i} \cdot p)$, the algorithm assigns 2^{-i-1} products to a buyer with price no less than $2^{-i} \cdot p$. For any sequence of buyers, the total amount of products assign to buyers is at most $1/2 + 1/4 + \cdots < 1$. The whole product is sufficient to be assigned to all buyers according to the algorithm.

The expected competitive ratio is

$$\begin{aligned}
\mathrm{E}[\frac{OPT}{ALG}] &= \int_1^{+\infty} \frac{x}{ALG(x)} dF(x) \\
&= (\int_1^{2^{-k}\cdot p} + \sum_{-k}^{-1} \int_{2^i \cdot p}^{2^{i+1}\cdot p} + \sum_{0}^{+\infty} \int_{2^i \cdot p}^{2^{i+1}\cdot p}) \frac{x}{ALG(x)} dF(x) \\
&\leq (\sum_{-k-1}^{-1} \int_{2^i \cdot p}^{2^{i+1}\cdot p} + \sum_{0}^{+\infty} \int_{2^i \cdot p}^{2^{i+1}\cdot p}) \frac{x}{ALG(x)} dF(x)
\end{aligned}$$

The above formula has two parts and we analyze them as follows.

(i) $-k-1 \leq i \leq -1$.
In this case, $ALG(x) \geq 2^{i-1} \cdot 2^i \cdot p$ while $x \leq 2^{i+1} \cdot p$. Thus,

$$\int_{2^i \cdot p}^{2^{i+1} \cdot p} \frac{x}{ALG(x)} dF(x) \leq 2^{2-i} \int_{2^i \cdot p}^{2^{i+1} \cdot p} dF(x) = 2^{2-i}(F(2^{i+1} \cdot p) - F(2^i \cdot p))$$

(ii) $i \geq 0$.
In this case, $ALG(x) \geq p/2$ while $x \leq 2^{i+1} \cdot p$. Thus,

$$\int_{2^i \cdot p}^{2^{i+1} \cdot p} \frac{x}{ALG(x)} dF(x) \leq 2^{i+2} \int_{2^i \cdot p}^{2^{i+1} \cdot p} dF(x) = 2^{i+2}(F(2^{i+1} \cdot p) - F(2^i \cdot p))$$

From Lemma 1, if $i \geq 0$, we have $2^i \cdot p(1-F(2^i \cdot p)) > 2^{i+1} \cdot p(1-F(2^{i+1} \cdot p))$. Thus, $1-F(2^{i+1} \cdot p) < (1-F(2^i \cdot p))/2$ and $F(2^{i+1} \cdot p)-F(2^i \cdot p) > (1-F(2^i \cdot p))/2$. Let $(1-F(2^{i+1} \cdot p) = (1-F(2^i \cdot p) \cdot \delta_i$ and $F(2^{i+1} \cdot p)-F(2^i \cdot p) = (1-F(2^i \cdot p)) \cdot \gamma_i$, where $\delta_i < 1/2$, $\gamma_i > 1/2$ and $\delta_i + \gamma_i = 1$.

From Lemma 3, $\frac{1-F(2x)}{1-F(x)}$ is monotone non-increasing when $x > p$, thus, δ_i is monotone non-increasing and γ_i is monotone non-decreasing when i increasing.

Thus, if $i \geq 0$,

$$\begin{aligned}
\int_{2^i \cdot p}^{2^{i+1} \cdot p} \frac{x}{ALG(x)} dF(x) &\leq 2^{i+2}(F(2^{i+1} \cdot p) - F(2^i \cdot p)) \\
&= 2^{i+2} \cdot (1 - F(2^i \cdot p)) \cdot \gamma_i \\
&= 2^{i+2} \cdot (1 - F(p)) \cdot \prod_{k=0}^{i-1} \delta_k \cdot \gamma_i \\
&\leq 2^{i+2} \cdot (1 - F(p)) \cdot \delta_0^i \\
&= 4 \cdot (1 - F(p)) \cdot (2\delta_0)^i
\end{aligned}$$

$$\begin{aligned}
\int_{p}^{+\infty} \frac{x}{ALG(x)} dF(x) &\leq 4 \cdot (1 - F(p)) \cdot \sum_i (2\delta_0)^i \\
&= \frac{4 \cdot (1 - F(p))}{1 - 2\delta_0}
\end{aligned} \tag{1}$$

From Lemma 1, if $i \leq -1$, we have $2^i \cdot p(1-F(2^i \cdot p)) < 2^{i+1} \cdot p(1-F(2^{i+1} \cdot p))$. Thus, $1-F(2^{i+1} \cdot p) > (1-F(2^i \cdot p))/2$ and $F(2^{i+1} \cdot p)-F(2^i \cdot p) < (1-F(2^i \cdot p))/2$. Let $1-F(2^{i+1} \cdot p) = (1-F(2^i \cdot p) \cdot \mu_i$ and $F(2^{i+1} \cdot p)-F(2^i \cdot p) = (1-F(2^i \cdot p)) \cdot \nu_i$, where $\mu_i > 1/2$, $\nu_i < 1/2$ and $\mu_i + \nu_i = 1$.

From Lemma 3, $\frac{1-F(x)}{1-F(2x)}$ is monotone non-decreasing when $2x < p$, and thus, μ_i is monotone non-decreasing and ν_i is monotone non-increasing when i increases.

Since

$$\begin{aligned} F(2^{i+2} \cdot p) - F(2^{i+1} \cdot p) &= (1 - F(2^{i+1} \cdot p) \cdot \nu_{i+1} \\ &= (1 - F(2^i \cdot p)) \cdot \mu_i \cdot \nu_{i+1} \\ &= (F(2^{i+1} \cdot p) - F(2^i \cdot p)) \cdot \mu_i \cdot \nu_{i+1}/\nu_i. \end{aligned}$$

We have

$$F(2^{i+1} \cdot p) - F(2^i \cdot p) = (F(2^{i+2} \cdot p) - F(2^{i+1} \cdot p)) \cdot \frac{\nu_i}{\mu_i \cdot \nu_{i+1}}.$$

Thus, if $i \leq -1$,

$$\begin{aligned} \int_{2^i \cdot p}^{2^{i+1} \cdot p} \frac{x}{ALG(x)} dF(x) &\leq 2^{2-i}(F(2^{i+1} \cdot p) - F(2^i \cdot p)) \\ &= 2^{2-i} \cdot (F(2^{i+2} \cdot p) - F(2^{i+1} \cdot p)) \cdot \frac{\nu_i}{\mu_i \cdot \nu_{i+1}} \\ &= 2^{2-i} \cdot (F(p) - F(p/2)) \cdot \frac{\nu_i}{\nu_0} \cdot \frac{1}{\prod_{k=i}^{0} \mu_k} \\ &\leq 8 \cdot (F(p) - F(p/2)) \cdot \frac{1}{\prod_{k=i}^{0} 2\mu_k} \\ &\leq 8 \cdot (F(p) - F(p/2)) \cdot (\frac{1}{2\mu_0})^{i+1} \end{aligned}$$

Therefore,

$$\begin{aligned} \int_{1}^{p} \frac{x}{ALG(x)} dF(x) &\leq 8 \cdot (F(p) - F(p/2)) \cdot \sum_i (\frac{1}{2\mu_0})^i \qquad (2) \\ &= \frac{8 \cdot (F(p) - F(p/2))}{1 - 1/(2\mu_0)} \end{aligned}$$

Combining the inequalities (1) and (2), we can say that the excepted competitive ratio of the algorithm is

$$\mathrm{E}[\frac{OPT}{ALG}] \leq \frac{4 \cdot (1 - F(p))}{1 - 2\delta_0} + \frac{8 \cdot (F(p) - F(p/2))}{1 - 1/(2\mu_0)} = O(1).$$

□

4 Distribution on The Highest Price of Each Buyer

In the previous part, we study the case that the distribution is on the highest price among all buyers. Now we assume that the distribution on the price of each buyer is known in advance, and the distribution on the buyers is under the i.i.d. assumption. We also assume that the number of buyers is bounded by n. Otherwise, even for a distribution with a very tiny value in some high price, the adversary can force the probability of the high price to be close to 1 by sending sufficiently large number of buyers.

Formally speaking, there are at most n buyers who will come to the seller to buy products; the price of each buyer is drawn from the accumulated distribution $F(x)$ with the density function $f(x)$, where $f(x)$ is derivable.

For this variant, if the distribution of the highest price among all buyers also satisfies the MHR property, the algorithms in Section 3 can be implemented. This gives us a heuristic to reduce this variant to the previous one. Let $\tilde{F}$(x) and $\tilde{f}(x)$ be the accumulated distribution and density function on the highest price among all buyers, thus, $\tilde{F}(x) = F^n(x)$ and $\tilde{f}(x) = nF^{n-1}(x)f(x)$, respectively.

Lemma 4. *If $f(x)$ satisfies the monotone hazard rate, then $\tilde{f}(x)$ also satisfies the monotone hazard rate.*

Proof. If $f(x)$ satisfies the monotone hazard rate (MHR), i.e., $\frac{1-F(x)}{f(x)}$ is non-increasing, we have $(\frac{1-F(x)}{f(x)})' \leq 0$, thus, $f'(x) \leq \frac{f^2(x)}{F(x)-1}$. Now we consider $\frac{1-\tilde{F}(x)}{\tilde{f}(x)}$. If $(\frac{1-\tilde{F}(x)}{\tilde{f}(x)})' \leq 0$, this lemma is true.

$$
\begin{aligned}
&(\frac{1-\tilde{F}(x)}{\tilde{f}(x)})' \\
&= (\frac{1-F^n(x)}{nF^{n-1}(x)f(x)})' \\
&= \frac{-(nF^{n-1}(x)f(x))^2 - (1-F^n(x))[n(n-1)F^{n-2}(x)f^2(x) + nF^{n-1}(x)f'(x)]}{(nF^{n-1}(x)f(x))^2} \\
&= \frac{-nF^n(x)f^2(x) - (n-1)f^2(x) - F(x)f'(x) + (n-1)F^n(x)f^2(x) + F^{n+1}(x)f'(x)}{nF^n(x)f^2(x)} \\
&= \frac{-F^n(x)f^2(x) - (n-1)f^2(x) + (F^{n+1}(x) - F(x))f'(x)}{nF^n(x)f^2(x)} \\
&\leq \frac{-F^n(x)f^2(x) - (n-1)f^2(x) + (F^{n+1}(x) - F(x))\frac{f^2(x)}{F(x)-1}}{nF^n(x)f^2(x)} \\
&= \frac{-F^{n+1}(x) + F^n(x) - (n-1)F(x) + (n-1) + F^{n+1}(x) - F(x)}{nF^n(x)(F(x)-1)} \\
&= \frac{(F(x)-1)(F^{n-1}(x)-1) - (n-1)(F(x)-1) + F^{n-1}(x) - 1}{nF^n(x)(F(x)-1)} \\
&= \frac{F^{n-1}(x) - 1 - (n-1) + F^{n-2}(x) + F^{n-3}(x) + \cdots + 1}{nF^n(x)} \\
&= \frac{F^{n-1}(x) + F^{n-2}(x) + \cdots + F(x) - (n-1)}{nF^n(x)} \\
&\leq 0
\end{aligned}
$$

Therefore, $\tilde{f}(x)$ also satisfies the monotone hazard rate. □

Since $\tilde{F}(x)$ and $\tilde{f}(x)$ satisfy the monotone hazard rate, Algorithm 1 and Algorithm 2 can be used to handle this variant. Thus, we have the following conclusion.

Theorem 3. *In the unbounded one-way trading problem, if the number of buyers is bounded, the distribution on price of each buyer is i.i.d. and satisfies the monotone hazard rate, online algorithms with constant competitive ratios can be obtained under the measures of* $\mathrm{E}[OPT/ALG]$ *and* $\mathrm{E}[OPT]/\mathrm{E}[ALG]$*.*

5 Concluding Remark

Design selling mechanisms to maximize the seller's revenue is well-studied in the field of economy whereas related research in theoretical computer science is relatively more recent and ongoing. Many variants of the problem have been found to be computationally difficult when cast in a realistic setting. The challenge has been to identify special cases for which a solution can be efficiently computed while keeping their relevance to real-life situations. Traditional worst case analyses in which the algorithm designer usually knows nothing about the future may not match the reality well. Average case analysis of the expected ratio is a direct measure of performance. This paper is an attempt to model the real case where the seller has some partial information about the buyers. For future research, it may be worthwhile to determine which information is critical and how to fully utilize the partial information to design selling strategies.

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