Interesting Problems for Cardinal Direction Relations: Composition and Consistency

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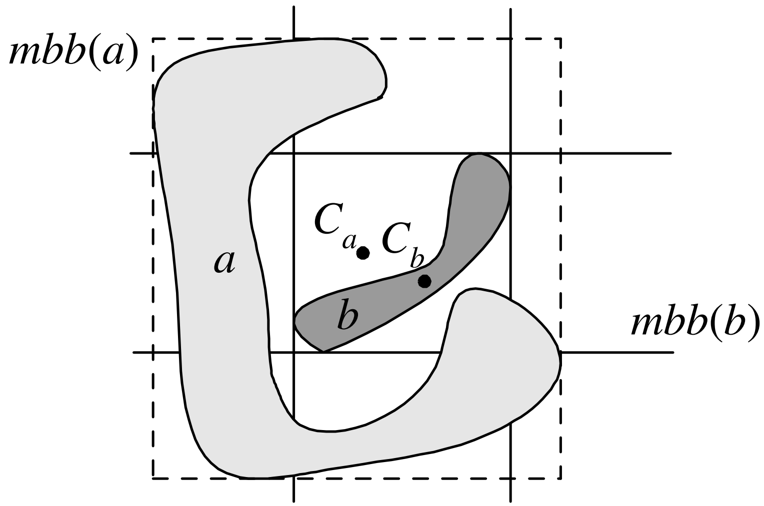
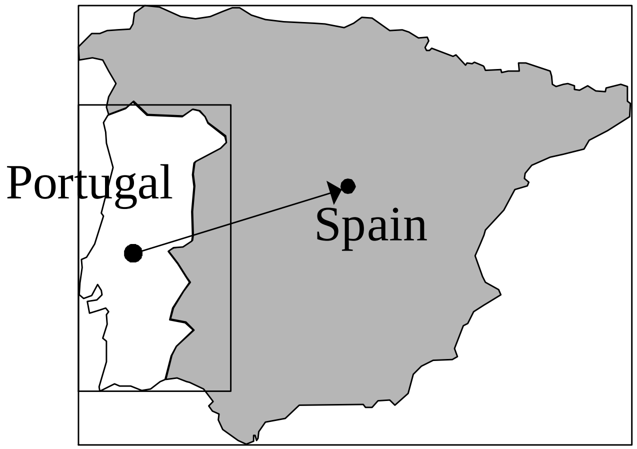
Talk Overview

- Models of qualitative spatial information
- Formal definition of cardinal direction relations
- The composition problem
- The consistency problem

Models of Qualitative Spatial Information

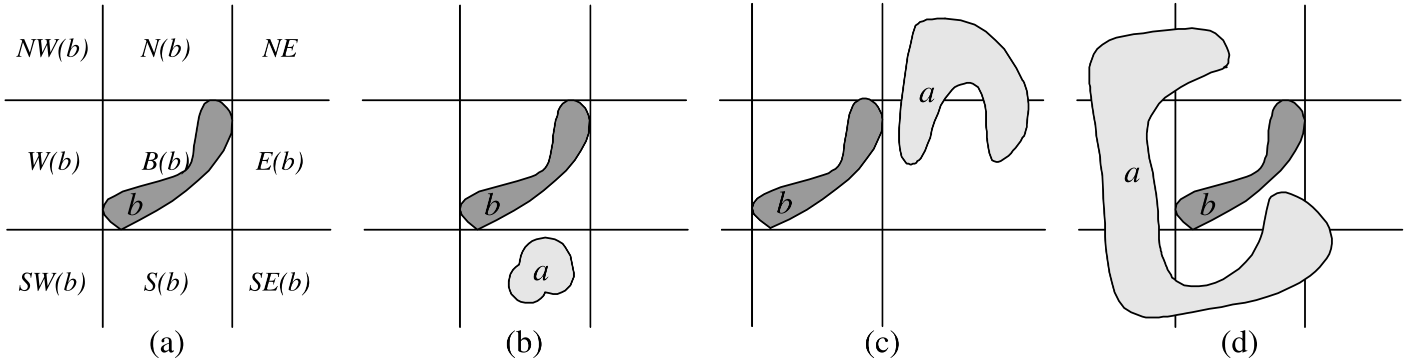
- **Models of spatial relations:** Directional relations [Ligozat98, Frank96], Topological relations [Egenhofer91], Distance relations [Frank92]





- **Approaches:** Point-based models [Frank96, Ligozat98, Freksa92], Minimum bounding box-based models [Papadias94, Adbelmoty94, Mukerjee+Joe90]

Cardinal Direction Relations



Single-tile

- In (b): $a\ S\ b$

Multi-tile

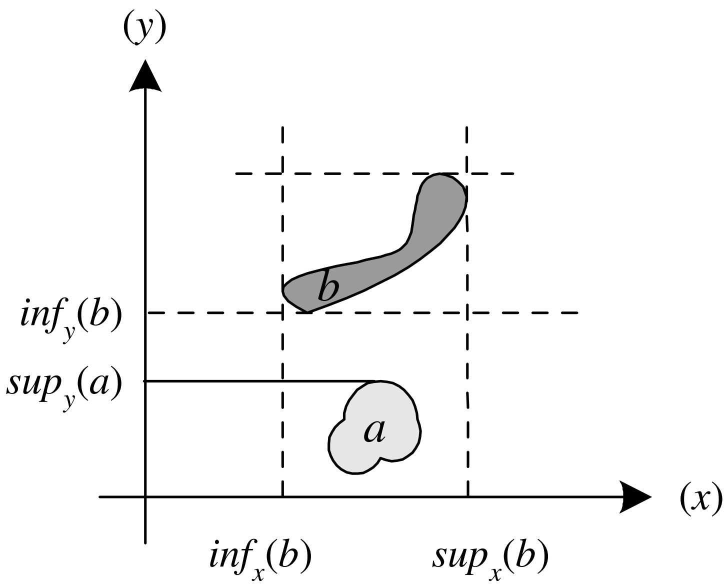
- In (c): $a\ NE{:}E\ b$
- In (d): $a\ B{:}S{:}SW{:}W{:}NW{:}N{:}E{:}SE\ b$

Presented in [Goyal and Egenhofer00, Skiadopoulos and Koubarakis01].

Defining Cardinal Direction Relations Formally (Single-tile)

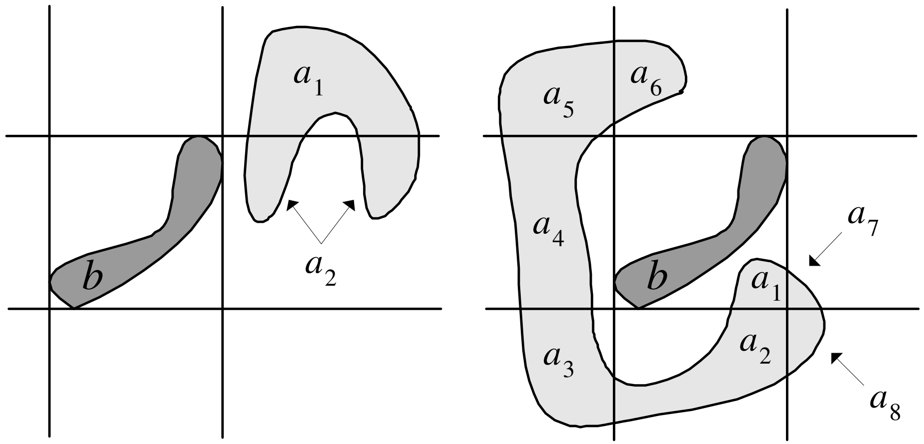
- Single-tile cardinal direction relations: B, S, SW, W, NW, N, NE, E and SE

$a\ S\ b$ iff $sup_y(a) \leq inf_y(b)$, $inf_x(b) \leq inf_x(a)$ and $sup_x(a) \leq sup_x(b)$



Defining Cardinal Direction Relations Formally (Multi-tile)

- a $NE{:}E$ b iff there exist regions a_1 and a_2 such that $a = a_1 \cup a_2$, a_1 NE b and a_2 E b



(a) (b)

- a $B{:}S{:}SW{:}W{:}NW{:}N{:}SE{:}E$ b iff there exist regions $a_1, \ldots, a_8$ such that $a = a_1 \cup a_2 \cup a_3 \cup a_4 \cup a_5 \cup a_6 \cup a_7 \cup a_8$, a_1 B b, a_2 S b, a_3 SW b, a_4 W b, a_5 NW b, a_6 N b, a_7 SE b and a_8 E b

Formal Definition: Multi-tile Relations (Cont'd)

General case:

- If $R_1, \ldots, R_k$ are single-tile cardinal directions relations then

$$a \; R_1{:}\cdots{:}R_k \; b \text{ iff there exist regions } a_1, \ldots, a_k \text{ such that}$$

$$\underbrace{a_1 \; R_1 \; b, \; \ldots, \; a_k \; R_k \; b}_{\text{Single-tile cardinal direction constraints}} \quad \text{and} \quad \underbrace{a = a_1 \cup \cdots \cup a_k}_{\text{Set-union constraints}}$$

Interesting problems

- Composition
  - Given that $a\ R_1\ b$ and $b\ R_2\ c$ holds, what is the relations between a and c?
  - E.g., from $a\ N\ b$ and $b\ N\ c$ we have that $a\ N\ C$ holds
- Consistency
  - C a set of constrains in variables $\bar{a}$. Is there an assignment of regions to the variables of $\bar{a}$ that satisfies all constrains of C?
  - E.g., set $\{a\ N\ b,\ b\ N\ c,\ a\ S\ b\}$ is inconsistent
- Use
  - Mechanism for inferring new spatial knowledge
  - Detecting inconsistencies
  - Preprocessing spatial queries

Composition of two cardinal direction relations

First problem

Composition (Consistency-based)

- Let R_1 and R_2 be binary relations. The **consistency-based composition** of relations R_1 and R_2, denoted by $R_1 \circ R_2$, is another binary relation which satisfies the following. $R_1 \circ R_2$ contains **all** relations $Q \in \mathcal{D}$ such that there exist regions $a, b, c \in REG$ such that $a \; R_1 \; b$, $b \; R_2 \; c$ and $a \; Q \; c$ hold

- For the rest of the talk composition refers to consistency-based composition

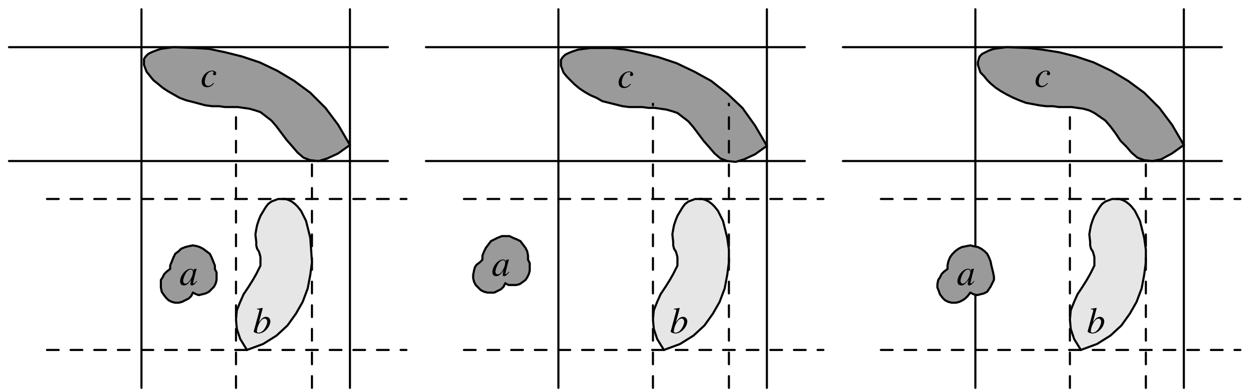
Composition Cases

- Single-tile with Single-tile
  - Transitivity table
- Single-tile with Multi-tile
  - Theorem
- Multi-tile with Multi-tile
  - Reduced to the composition of single with multi-tile

Single-tile with Single-tile

- **Example**

$$W \circ S \overset{(Table)}{=} \delta(S, SW) = \{S,\ SW,\ S{:}SW\}$$



Single-tile with Multi-tile

- Let R_1 be a single-tile and $R_2 = R_{21}{:}\cdots{:}R_{2l}$ be a multi-tile cardinal direction relation. Does

$$R_1 \circ (R_{21}{:}\cdots{:}R_{2l}) = \delta(R_1 \circ R_{21}, \ldots, R_1 \circ R_{2l})$$

  hold?

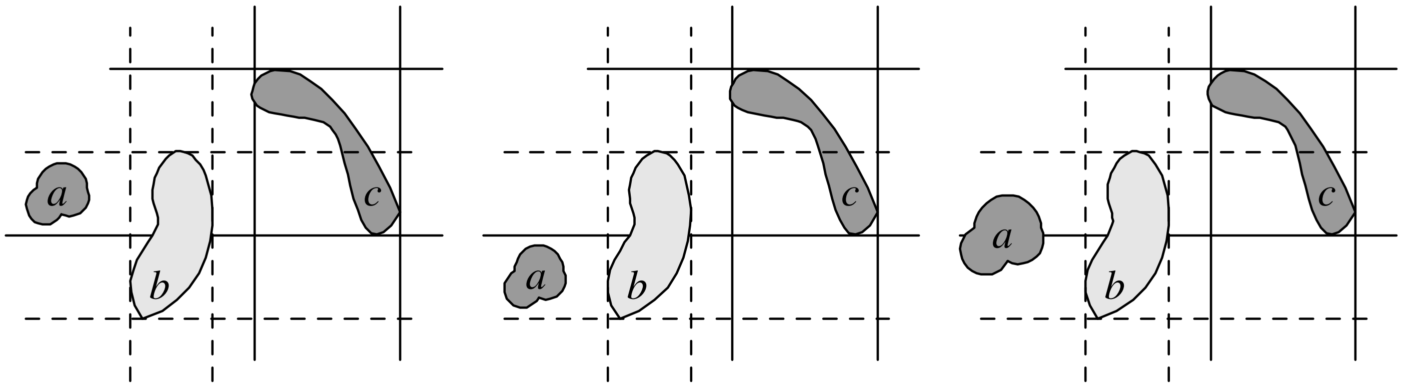
- Only for a very small fraction (11%) of the possible pairs!

Single-tile with Multi-tile

- **Example**

  Composing W and $SW{:}W$

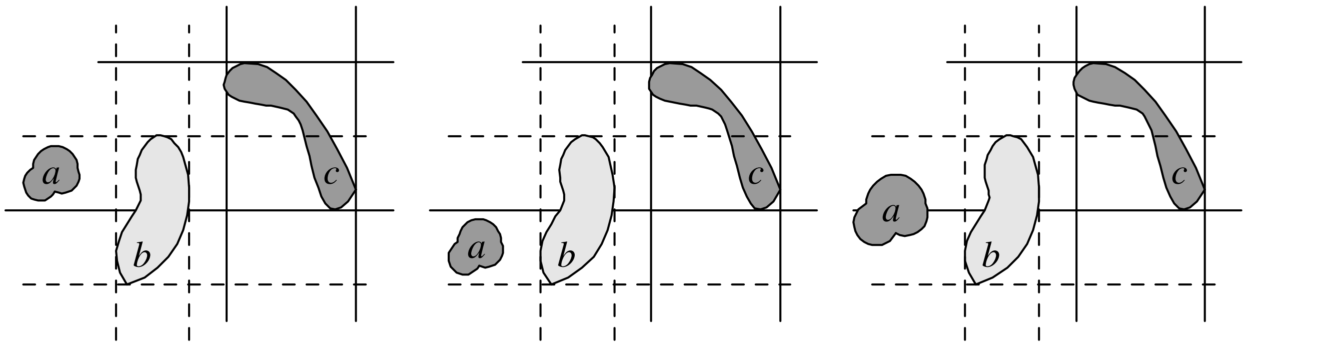
  $W \circ SW{:}W = \delta(W \circ SW, W \circ W) = \delta(SW, W) = \{SW,\ W,\ SW{:}W\}$



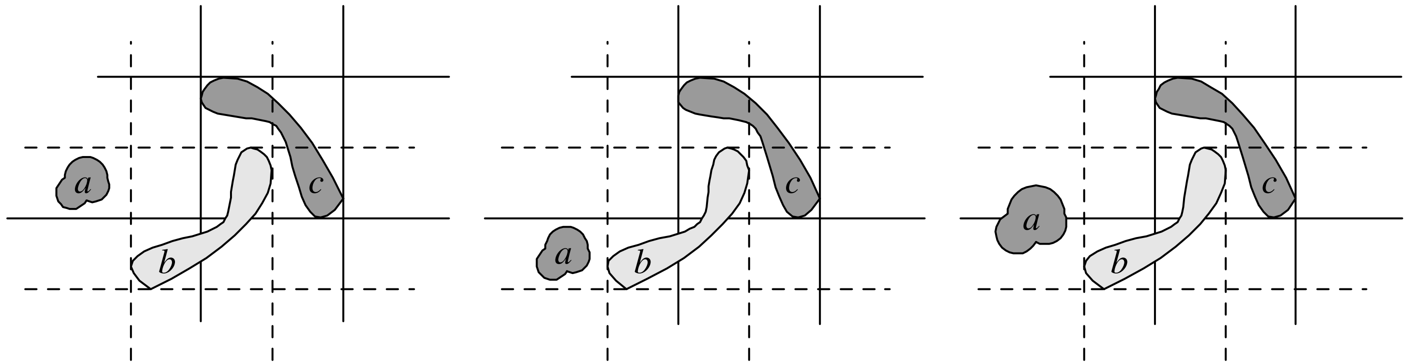
Single-tile with Multi-tile

- **Example**

$$W \circ SW{:}W = \{W,\ SW,\ SW{:}W\}$$



$$W \circ B{:}S{:}SW = \{W,\ SW,\ SW{:}W\}$$



Single-tile with Multi-tile

- **Theorem**

  Let R_1 be an single-tile cardinal direction relation and R_2 be a multi-tile cardinal direction relation. Then

$$R_1 \circ R_2 = R_1 \circ \mathcal{M}ost(R_1, \mathcal{B}r(R_2)).$$

- $R_1 \circ \mathcal{M}ost(R_1, \mathcal{B}r(R_2))$ is always in the well-behaved 11%!
- $\mathcal{B}r(R)$:

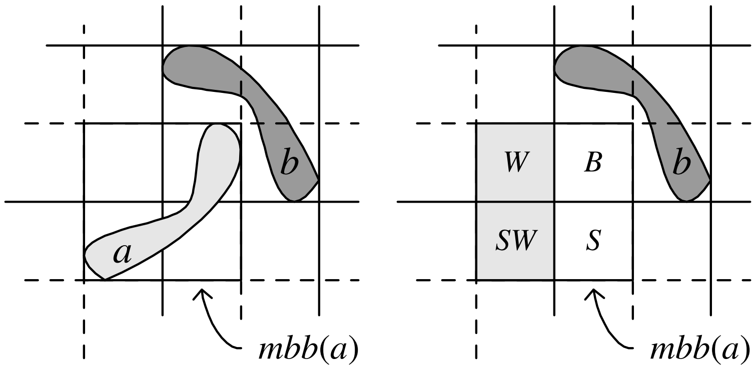
  if $a\ R\ b$ then $mbb(a)\ \mathcal{B}r(R)\ b$
- $\mathcal{M}ost(R_1, R_2)$:

  is the relation formed by the R_1-most tiles of R_2

Single-tile with Multi-tile

- **Example**

  From $W \circ B{:}S{:}SW$ to $W \circ SW{:}W$



  - $\mathcal{B}r(B{:}S{:}SW) = B{:}S{:}SW{:}W.$
  - $\mathcal{M}ost(W, \mathcal{B}r(B{:}S{:}SW)) = SW{:}W$

- Thus $W \circ B{:}S{:}SW = W \circ SW{:}W$

Multi-tile with Multi-tile

- **Theorem**

  Let $R_1 = R_{11}:\cdots:R_{1k}$ and R_2 be multi-tile cardinal direction relations, where $R_{11}, \ldots, R_{1k}$ are single-tile cardinal direction relations. Then

$$R_1 \circ R_2 = \{Q \in \mathcal{D} : \ (\exists s_1, \ldots, s_k)\ (Q = \mathit{tile\text{-}union}(s_1, \ldots, s_k) \wedge s_1 \in R_{11} \circ R_2 \wedge \cdots \wedge s_k \in R_{1k} \circ R_2)\}.$$

- $\mathit{tile\text{-}union}(R_1, R_2)$ is the multi-tile cardinal direction relation that consists of all the tiles in R_1 and R_2

In the paper [SK04] ...

- Formal definitions of the cardinal direction relations model
- Transitivity table of single-tile relations
- Formal proofs for all Algorithms/Theorems
- [SK04] S. Skiadopoulos and M. Koubarakis. Composing Cardinal Direction Relations. *Artificial Intelligence*, 152(2):143–171, 2004.

Summary – Composition

- Formal definition of cardinal direction relations model
- Discussion of the composition operation
  - Considered several classes of cardinal direction relations and gave algorithms for composition
  - The above algorithms for composition also hold for connected regions, disconnected regions and regions with holes, points and lines

Consistency of a set of cardinal direction constraints

Second problem

Outline of the Consistency Algorithm

Algorithm CONSISTENCY

Input: A set C of **single/multi-tile** cardinal direction constraints

Output: 'Consistent' if C is consistent; 'Inconsistent' otherwise

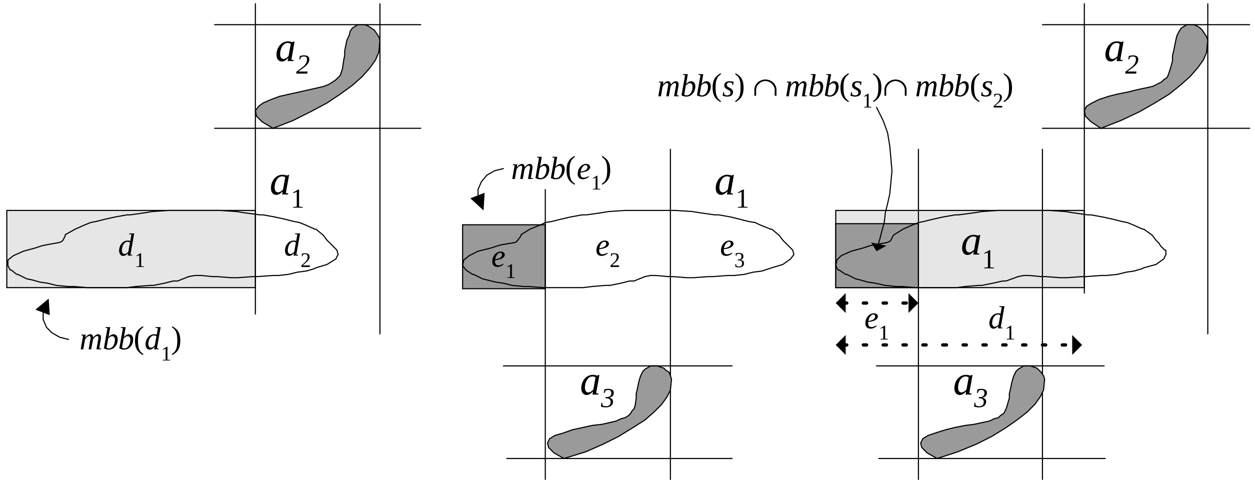
- **Step 1** Considers each constraint c in C and maps the single-tile cardinal direction constraints into a set of **order constraints** O
- **Step 2**
  Finds a **"maximal"** solution of the set of order constraints O
- **Step 3**
  Checks whether the above solution satisfies the set-union constraints

$a \ R_1{:}\cdots{:}R_k \ b$ iff there exist regions $a_1, \ldots, a_k$ such that

$$\underbrace{a_1 \ R_1 \ b, \ldots, a_k \ R_k \ b}_{\text{Single-tile cardinal direction constraints}} \quad \text{and} \quad \underbrace{a = a_1 \cup \cdots \cup a_k}_{\text{Set-union constraints}}$$

Running Example

$$C = \{a_1 \ S{:}SW \ a_2, \ a_1 \ NW{:}N{:}NE \ a_3\}$$



Step 1

Considers the definition of every cardinal direction constraint $a_i \; R_1{:}\cdots{:}R_k \; a_j$ in C

$$a \; R_1{:}\cdots{:}R_k \; b \text{ iff there exist regions } a_1, \ldots, a_k \text{ such that } \underbrace{a_1 \; R_1 \; b, \ldots, a_k \; R_k \; b}_{\text{Single-tile cardinal direction constraints}} \quad \text{and} \quad \underbrace{a = a_1 \cup \cdots \cup a_k}_{\text{Set-union constraints}}$$

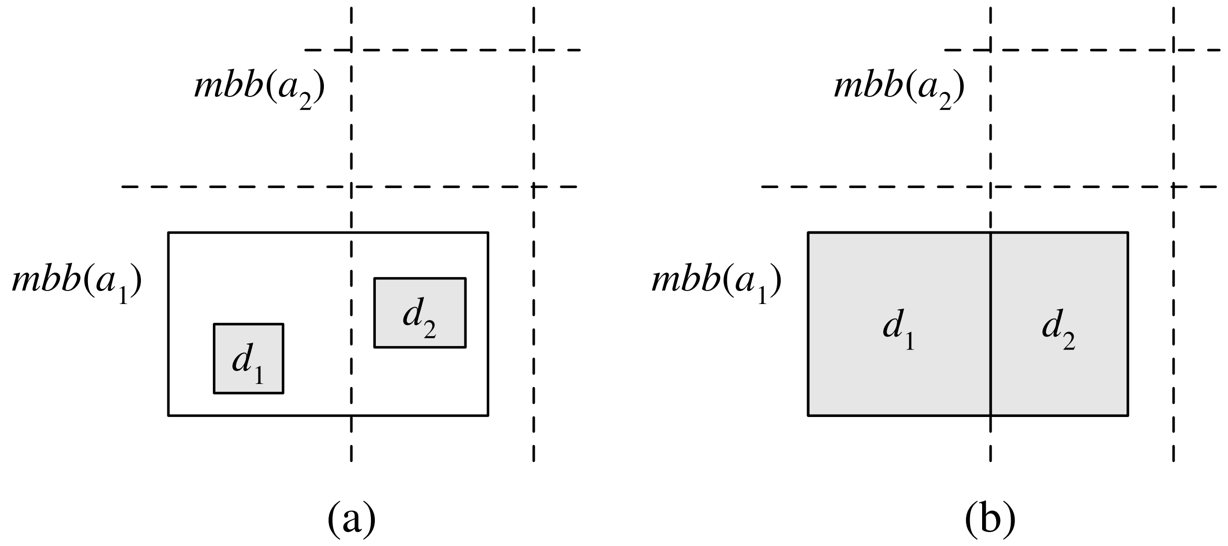
- Introduces new region variables
- Introduces into O **order constraints** encoding all single-tile cardinal direction relations
- Introduce into O the obvious order constraints relating the endpoints
- Introduce into O additional order constraints that establish the **strictest possible** relation between the endpoints

Step 2

- Finds a solution of set O [van Beek92, Delgrande et al99].

  If no solution exists then return 'Inconsistent'

  else extend the regions that correspond to component variables (find a **"maximal" solution**)



Step 3

- Checks if the “maximal” solution of O satisfies a special constraint NTB which is **weaker** than the set union constraints

  If constraint NTB is satisfied then return ‘Consistent’
  else return ‘Inconsistent’

Results

- **Theorem 1**

  Let C be a set of single/multi-tile cardinal direction constraints. Algorithm Consistency correctly decides whether C is consistent

  – The proof is long

- **Theorem 2**

  Checking the consistency of a set of **single/multi-tile** cardinal direction constraints in n variables can be done using Algorithm Consistency in $\mathcal{O}(n^5)$ time

  – Easy to see

Results (Cont'd)

- **Theorem 3**

  Checking the consistency of an **unrestricted** set of cardinal direction constraints is NP-complete

  – A reduction that is based on the interval case

- An extension of the model that handles points and lines

Discussion

- The above algorithm can be applied for
  - Disconnected extended regions
  - Disconnected arbitrary region in $\Re^2$ (points – lines)
- We believe that it also works for connected regions but we do not have a proof
- More details and proofs
  - S. Skiadopoulos and M. Koubarakis. On the Consistency of Cardinal Directions Constraints. *Artificial Intelligence*, 163(1):91–135, 2005.

Summary

- Formal Definition of Cardinal Direction Relations Model
- Discussion of the Composition Operation
  - Considered several classes of cardinal direction relations and gave algorithms for composition
- Consistency Problem for Cardinal Direction Constraints
  - Present the first algorithm for this problem and prove its correctness
  - Consistency checking of a set of **single/multi-tile** cardinal direction constraints can be done in $\mathcal{O}(n^5)$
  - Consistency checking of an **unrestricted** set of cardinal direction is NP-complete