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Continuous Detection of the Variations of the Intersection Curves of Two Moving Quadrics in 3-Dimensional Projective Space

Xiaohong Jia[a,*], Wenping Wang[b], Yi-King Choi[b], Bernard Mourrain[c], Changhe Tu[d]

[a]*KLMM, NCMIS, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing, China*
[b]*The University of Hong Kong, Pokfulam Road, Hongkong, China*
[c]*GALAAD, INRIA Méditerranée, 2004 route des lucioles, 06902 Sophia-Antipolis, France*
[d]*ShanDong University*

Abstract

We propose a symbolic algorithm for detecting the variations in the topological and algebraic properties of the intersection curve of two moving quadrics in $\mathbb{PR}^3$(real projective 3-space). The core of our algorithm computes all the instants when the intersection curve of two moving quadrics changes type using resultants and Jordan forms. These instants partition the time axis into intervals within which the type of the intersection curve of the two moving quadrics can be determined by the method proposed in [Tu et al. (2009)]. Examples are provided to illustrate our algorithm.

Keywords: intersection curve; moving quadrics; signature sequence; index function; Jordan form

1. Introduction

1.1. Background

Quadrics are the simplest curve surfaces which have both concise algebraic representations and elegant geometric properties. Hence they have been widely used in CAD/CAM and industrial manufacture, where 3D shapes are frequently defined by piecewise quadrics [Wang (2002), Yan et al. (2012)]. The intersection curve of two quadrics (QSIC) has attracted particular attention since it contributes to the boundary detection of 3D shapes defined by quadric patches in geometric modeling or industry design [Levin (1979), Wang et al. (2003), Dupont et al. (2008a,b,c), Tu et al. (2009)].

There is plenty of work on the classification of the intersection curve of two quadrics. In algebraic geometry, QSIC morphology is usually classified in $\mathbb{PC}^3$, complex projective 3-space. For example in [Bromwich (1906)]. Bromwich accomplishes this classification by means of the Segre characteristic, which is defined by the multiplicities of roots of the characteristic polynomial of the two given quadrics, without distinguishing between real and imaginary roots. Hence in $\mathbb{PR}^3$, real projective 3-space, different QSICs might correspond

*Corresponding author
Email address: `xhjia@amss.ac.cn(X. Jia)`, `wenping@cs.hku.hk(W. Wang)`, `ykchoi@cs.hku.hk(Y.-K. Choi)`, `mourrain@sophia.inria.fr (B. Mourrain)`, `chtu@sdu.edu.cn(C. Tu)` (Changhe Tu)

to the same Segre characteristic and therefore cannot be further distinguished by [Bromwich (1906)]. [Tu et al. (2009)] resolved this problem by directly classifying the QSICs in $\mathbb{PR}^3$ using signature sequences, a technique finer than the Segre characteristics, which further takes the real roots and the Jordan forms associated with these real roots of the characteristic polynomial into consideration. Moreover, their classification distinguishes between the QSICs both by their topological properties and the algebraic properties, which is slightly finer than morphology. As for computation, [Tu et al. (2009)] also provides an algorithm with rational arithmetic for computing the signature sequence and then determining the type of the QSIC. For other literature on computation of the QSICs, see [Levin (1979), Dupont et al. (2008a,b,c)] for examples.

To our knowledge, there is almost no work on the continuous detection of the QSIC variations of two moving quadrics. The closest existing work is on continuous collision detection of two moving ellipsoids, which aims at detecting the instants when the two moving quadrics touch each other, i.e., when the intersection curve of the two moving quadrics turns from imaginary to real, or conversely. For related work see [Ju et al. (2001); Rimon and Boyd (1997); Shiang et al. (2000); Wang et al. (2004); Choi et al. (2006, 2009); Jia et al. (2011)].

The objects considered in our paper are two quadrics that are moving or deforming in $\mathbb{PR}^3$. Our target is to detect the type variations of their QSIC in $\mathbb{PR}^3$. Moreover, we consider the QSIC from a slightly finer aspect than the common concept of morphology, i.e., we distinguish the QSICs by their topological properties and algebraic properties, including singularities, the number of components, and the degree of each of the irreducible components. For example, we distinguish a simple real loop from a double loop; we distinguish a simple real loop from a real loop with a cusp; we also distinguish a non-degenerate QSIC with two disconnected component from a reducible QSIC with two disconnected conics. Our approach to distinguishing the type of the QSICs agrees exactly with that of [Tu et al. (2009)].

Our two moving quadrics are given by $\mathcal{A} : X^T A(t) X = 0$ and $\mathcal{B} : X^T B(t) X = 0$, where $X = (x, y, z, w)^T \in \mathbb{PR}^3$ and $A(t), B(t)$ are 4×4 matrices whose elements are real functions in t. The *characteristic function* associated with $\mathcal{A}$ and $\mathcal{B}$ is defined by

$$f(\lambda; t) = \det(\lambda A(t) - B(t)), \tag{1}$$

where $\lambda \in \mathbb{R}$, and hence $f(\lambda; t)$ is either a bivariate function in λ, t of degree at most four in λ, or a univariate polynomial of degree at most four in λ or vanishes identically. The moving quadric pencil $\lambda A(t) - B(t)$ is said to be *non-degenerate* if $f(\lambda; t) \not\equiv 0$, otherwise the pencil is said to be *degenerate*. The moving quadric pencil is degenerate if and only if at an arbitrary instant t the two moving quadrics $\mathcal{A}$ and $\mathcal{B}$ are two singular quadrics sharing a singular point or a double line. For example, $\mathcal{A}(t)$ and $\mathcal{B}(t)$ are two cones sharing the same vertex but both rotating in their own way around the vertex. Analyzing degenerate pencils is a relatively simple task and can be treated in a way quite similar to the analysis of non-degenerate pencils we are going to describe. Hence at the end of the paper we shall provide a degenerate example but throughout the rest of the paper we shall assume that $f(\lambda; t) \not\equiv 0$.

Our approach is twofold: First we detect all the discrete instants at which the QSIC changes, which is proved to be equivalent to detecting the instants when the Segre characteristic of the quadric pencil $\lambda A(t) - B(t)$ changes. This step is the core of the whole paper, in which we separately treat two different situations depending on whether the characteristic equation $f(\lambda; t)$ has a multiple factor in $\mathbb{R}[\lambda; t]$ or not. The candidate instants we compute for the QSIC change partition the time axis into time intervals. Then we determine the QSIC during each of these intervals by the technique proposed in [Tu et al. (2009)].

The paper is organized as follows. In Section 2 we provide preliminaries on the algebraic tools we shall use throughout the paper, i.e., the Jordan form of matrices, and the Segre characteristics and signature sequence of the pencil formed by two quadrics. Based on these concepts we briefly review the algorithm for deciding the QSIC type in $\mathbb{PR}^3$ provided by [Tu et al. (2009)]. In Section 3 we show how to detect the type variations of the QSICs and present the corresponding symbolic algorithm. We provide several examples in Section 4 to illustrate our algorithm. We conclude in Section 5 with possible future work.

2. Preliminaries

In this paper, all the quadrics are assumed to be in the real projective 3-space $\mathbb{PR}^3$. A quadric $\mathcal{A}$ in $\mathbb{PR}^3$ is represented by the equation $X^T AX = 0$, where $X = (x, y, z, w)^T$ are the homogeneous coordinates of the point $(\frac{x}{w}, \frac{y}{w}, \frac{z}{w})$ in $\mathbb{R}^3$, and A is a 4×4 matrix with elements in $\mathbb{R}$.

2.1. Jordan forms

Two matrices M and N from $\mathbb{R}^{k\times k}$ are said to be *similar* if there is a nonsingular matrix P such that $M = P^{-1}NP$. Each matrix $A \in \mathbb{R}^{k\times k}$ is simialr to its Jordan normal form, which shall be reviewed below.

Definition 2.1. *A $k \times k$ square matrix of the form*

$$M = \begin{pmatrix} \lambda & e & & \\ & \cdot & \cdot & \\ & & \cdot & e \\ & & & \lambda \end{pmatrix}_{k\times k}$$

is called a Jordan block of type I associated with λ if $\lambda \in \mathbb{R}$ and $e = 1$ for $k \geq 2$ or $M = (\lambda)$ with $\lambda \in \mathbb{R}$ for $k = 1$; M is called a Jordan form of type II associated with complex conjugate values $a \pm ib$ if

$$\lambda = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \quad a, b \in \mathbb{R}, b \neq 0 \text{ and } e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

for $k \geq 4$ *or*

$$M = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

for $k = 2$*, with* $a, b \in \mathbb{R}, b \neq 0$*.*

Definition 2.2. *For any matrix* $A \in \mathbb{R}^{k \times k}$*, there exists a quasi-diagonal matrix* J

$$J = \begin{pmatrix} C(\lambda_1) & & & \\ & C(\lambda_2) & & \\ & & \ddots & \\ & & & C(\lambda_k) \end{pmatrix}$$

similar to A*, where*

$$C(\lambda_i) = \begin{pmatrix} J_1^{(i)} & & & \\ & J_2^{(i)} & & \\ & & \ddots & \\ & & & J_{k_i}^{(i)} \end{pmatrix}$$

for which $J_1^{(i)}, \cdots, J_{k_i}^{(i)}$ *are all Jordan blocks (of type I or type II) associated with the same eigenvalue* λ_i *of the matrix* A*. The quasi-diagonal matrix* J *is called the Jordan normal form of the matrix* A*, and the blocks* $C(\lambda_i)$ *are called the full Jordan chain associated with the eigenvalue* λ_i*,* $i = 1, \cdots, k$*. The Jordan normal form is unique up to permutation of the Jordan blocks.*

The following two Lemmas repeatedly serve as the mathematical underpinnings of our main algorithm.

Lemma 2.1. *The minimal polynomial of the Jordan block*

$$M = \begin{pmatrix} \lambda_0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_0 \end{pmatrix}_{k \times k},$$

i.e, the monic polynomial $p(\lambda)$ *of least degree such that* $p(M) = 0$*, is* $(\lambda - \lambda_0)^k$*.*

Lemma 2.2. *The minimal polynomial of a quasi-diagonal matrix* J *is equal to the L.C.M. of the minimal polynomials of each diagonal block of* J*.*

2.2. Segre characteristic and index sequences

Two quadrics $\mathcal{A}, \mathcal{B}$ with equations $X^TAX = 0$ and $X^TBX = 0$ define a *quadric pencil* $\lambda A - B$, where $\lambda \in \mathbb{R}$. We define the *characteristic polynomial* associated with A and B by

$$f(\lambda) = \det(\lambda A - B).$$

The quadric pencil $\lambda A - B$ is said to be *degenerate* if $f(\lambda) \equiv 0$, otherwise the pencil is *non-degenerate*. The quadric pencil $\lambda A - B$ is degenerate if and only if $\mathcal{A}$ and $\mathcal{B}$ are two singular quadrics sharing a common point or a double line [Tu et al. (2009)]. Note that the eigenvalues of a symmetric matrix are all real. Hence for arbitrary $\lambda \in \mathbb{R}$, all the eigenvalues of the matrix $\lambda A - B$ are real.

Lemma 2.3. *The characteristic polynomial of the matrix $A^{-1}B$ and the polynomial $f(\lambda)$ have the same roots with the same multiplicities.*

Definition 2.3. *The Segre characteristic of the quadric pencil $\lambda A - B$ is the integer chain of the orders of the blocks in the Jordan normal form of the matrix $A^{-1}B$, with those integers corresponding to blocks containing the same eigenvalue bracketed together, and the number of distinct real eigenvalues of the matrix $A^{-1}B$ as the subscript. For example, if the Jordan form of the matrix $A^{-1}B$ is*

$$\begin{pmatrix} \alpha & 1 & & \\ & \alpha & & \\ & & \alpha & \\ & & & \beta \end{pmatrix}$$

where α, β are real numbers, the Segre characteristic of the quadric pencil $\lambda A - B$ is $[(21)1]_2$. In this example we also say that the Segre characteristic of $\lambda A - B$ associated with the root α is $[21]$, and the Segre characteristic of $\lambda A - B$ associated with the root β is $[1]$

Definition 2.4. *For a fixed λ, the index of the quadric pencil $\lambda A - B$, denoted by $\mathrm{Id}(\lambda)$, is the number of positive eigenvalues of the matrix $\lambda A - B$.*

Definition 2.5. *Let $\lambda_j, j = 1, \cdots, r$ be all the distinct real roots of $f(\lambda)$ in increasing order. Let $q_k,\ k = 1, \cdots, r-1$ be any real numbers separating λ_j, i.e.,*

$$-\infty < \lambda_1 < q_1 < \lambda_2 < \cdots < q_{r-1} < \lambda_r < \infty.$$

Let $s_j = \mathrm{Id}(q_j), j = 1, \cdots, r-1,\ s_0 = \mathrm{Id}(-\infty)$ and $s_r = \mathrm{Id}(\infty)$. Then the index sequence of the quadric pencil of $\lambda\mathcal{A} - \mathcal{B}$ is defined as

$$\langle s_0 \uparrow s_1 \uparrow \cdots \uparrow s_r \rangle$$

where each $\uparrow$ stands for the real root λ_i of $f(\lambda) = 0$ sequentially.

Since a real root λ_i could be multiple, and the Jordan chain associated to a multiple root λ_i of the matrix $\lambda_i A - B$ could have different forms, we shall distinguish between these cases. Instead of $\uparrow$ we use $|$ to denote a real root associated with a 1×1 Jordan block, and we use $\wr$ for p ($p \geq 2$) times in a row to denote a real root associated with a $p \times p$ Jordan block. For example, a real root with Segre characteristic [211] is denoted by $\wr\|$ in place of an $\uparrow$.

Example 2.1.

2.3. Review of Tu's algorithm

[Tu et al. (2009)] presents a classification of the QSICs in $\mathbb{PR}^3$, in terms of the same philosophy as ours in distinguishing the QSIC types, i.e, both algebraic properties and topological properties are considered, including singularities, number of components, and the degree of each irreducible component, as mentioned in the introduction. These authors also provide an algorithm for deciding the type of the QSIC based on this classification. The core of their algorithm computes the signature sequence (p321, [Tu et al. (2009)]) of the pencil of the two given quadrics, which is computationally more efficient than but theoretically equivalent to identifying the index sequence of the quadric pencil. Once the index sequence (or the signature sequence) of the quadric pencil is computed, together with the Segre characteristic one can find the corresponding QSIC type from Table 1,2,3 in [Tu et al. (2009)].

In this paper we shall not go deep into issues of signature sequences. Instead we prefer to analyze index sequences, because index sequences have a concise notation. They also appear in our main theorem (Theorem 3.2) on Segre characteristics, which play the most important role in detecting the instants where the QSIC changes, the core step of our approach. Once these instants are computed, readers are left to determine the QSIC during each of the intervals between these instants, and at this stage signature sequences are preferred because of efficiency. In the following example, our goal is to further explain the theory of index sequences, readers are referred to [Tu et al. (2009)] if additional computational efficiency is required.

Example 2.2. *Given two quadrics $\mathcal{A} : X^T AX = 0$ and $\mathcal{B} : X^T BX = 0$, where $X = (x, y, z, w)^T$ and*

$$A = \begin{pmatrix} 1 & \frac{1}{2} & & \\ \frac{1}{2} & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -\frac{1}{2} & & \\ -\frac{1}{2} & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}.$$

The characteristic polynomial associated with $\mathcal{A}$ and $\mathcal{B}$ is

$$f(\lambda) = \det(\lambda A - B) = -\frac{3}{4}\lambda^4 + \frac{5}{4}\lambda^3 - \frac{5}{4}\lambda + \frac{3}{4},$$

which has four real roots $\lambda_1 = -1, \lambda_2 = \frac{1}{3}, \lambda_3 = 1, \lambda_4 = 3$. *So the Segre characteristic associated with the quadric pencil* $\lambda A - B$ *is* $[1111]_4$. *From Table 1 in [Tu et al. (2009)], there are two possible index sequences as well as signature sequences corresponding to this Segre characteristic. Choosing* $s_0 = -2 < \lambda_1 < s_1 = 0 < \lambda_2 < s_2 = \frac{1}{2} < \lambda_3 < s_3 = 2 < \lambda_4 < s_4 = 4$, *and computing* $\mathrm{Id}(s_i), i = 1, 2, 3, 4$, *we find that the index sequence of the quadric pencil* $\lambda A - B$ *is* $\langle 1|2|3|2|3 \rangle$, *which is equivalent to* $\langle 1|2|1|2|3 \rangle$ *by the equivalence rules of index sequences listed on page 322 of [Tu et al. (2009)]. Therefore, according to Table 1 in [Tu et al. (2009)], the intersection curve of these two quadrics consists of two separate real loops.* $\square$

3. Continuous Detection of the topological variations of the intersection curve

In the following our two quadrics $\mathcal{A}, \mathcal{B}$ in $\mathbb{PR}^3$ are moving under affine motions, i.e, the homogeneous representations of these two moving quadrics is $X^T A(t) X = 0$ and $X^T B(t) X = 0$, where $A(t), B(t)$ are 4×4 matrices whose elements are in $\mathbb{R}(t)$, the real function field in t. We define the *characteristic function* associated with $\mathcal{A}$ and $\mathcal{B}$ by

$$\hat{f}(\lambda; t) = \det(\lambda A(t) - B(t)),$$

where $\lambda \in \mathbb{R}$. Clearly $f(\lambda; t) \in \mathbb{R}(t)[\lambda]$, hence can be taken as polynomials in λ whose coefficients are real functions in t. Suppose the leading coefficient of $\hat{f}(\lambda; t)$ with respect to λ to be $l(t)$. We define

$$f(\lambda; t) = \hat{f}(\lambda; t)/l(t)$$

as the *normalized characteristic function* associated with $\mathcal{A}$ and $\mathcal{B}$. Note that $f(\lambda; t)$ has no non-trivial factors in $\mathbb{R}(t)$.

Note that $f(\lambda; t)$ is either a bivariate function of degree at most four in λ or a univariate polynomial of degree at most four in λ, or vanishes identically. As mentioned before, throughout this paper we shall assume that $f(\lambda; t)$ does not vanish identically. The case that $f(\lambda; t)$ is a univariate polynomial in λ will be considered as a special case of a bivariate function whose coefficients in λ are constant functions in t, so will not be treated separately.

Before we start, observe that special care should be taken for those instants t_i that are real roots of $l(t) = 0$, when the quadric pencil $\lambda A(t_i) - B(t_i)$ is degenerate since $f(\lambda; t_0) \equiv 0$. We must also take care of those instants t_i that are roots of $l(t) = \infty$, when the quadric $A(t_0)$ or $B(t_0)$ changes to another type of quadric since some elements of the matrices have zero denominators, for example, a ellipsoid becomes a double plane if the length of some of its axes goes to $+\infty$. Therefore, at the very beginning we shall take these t_i as candidates when the QSIC might change.

3.1. Problem simplification

The outline of our approach to detecting the variations of the intersection curve $\mathcal{C}$ of the two moving quadrics will be:

1. Detect the instants $T = \{t_i\}_{i=1}^{n}$ across which the type of QSIC changes;
2. Set $t_0 = -\infty$ and $t_{n+1} = +\infty$, and let $\mathcal{I}_i = (t_i, t_{i+1}), i = 0, \cdots, n$ be the intervals partitioned by t_i; Arbitrarily choose real numbers $s_i, i = 1, \cdots, n+1$ such that $t_0 < s_1 < t_1 < s_2 < t_2 < \cdots < s_{n+1} < t_{n+1}$;
3. For each interval $\mathcal{I}_i, i = 1, \cdots, n$, determine the QSIC of $\lambda A(s_i) - B(s_i)$ by Algorithm 1 in [Tu et al. (2009)].
4. For each time instant t_i, determine the QSIC of $\lambda A(t_i) - B(t_i)$ by Algorithm 1 in [Tu et al. (2009)].

By [Tu et al. (2009)], a Segre characteristics and an index sequence together determines a QSIC. Hence intuitively we should first detect all the instants $t_i,\ i = 1, \cdots, m$ when the Segre characteristic of the quadric pencil $\lambda A(t) - B(t)$ changes. These t_i partition the time axis into intervals $t_i < t < t_{i+1}, i = 0, \cdots, m$. Then we have to further refine these intervals by checking the changes of the index sequences. However, the following theorem shows that *the latter step is unnecessary.* That is, *detecting the changes in the Segre characteristic is enough to decide the QSIC changes of the two moving quadrics.*

Lemma 3.1. *Let $a_j(t), 1 \leq j \leq n$, be continuous complex-valued functions defined on an interval $\mathcal{I}$. Then there exist continuous complex-valued functions $\lambda_1(t), \cdots, \lambda_n(t)$ which, for each $t \in \mathcal{I}$, are the roots of the polynomial equation $\lambda^n - a_1(t)\lambda^{n-1} + \cdots + (-1)^n a_n(t) = 0$.*

Theorem 3.2. *During a continuous time span of t, if the Segre characteristic of the pencil $\lambda A(t) - B(t)$ does not change, the index sequence of the quadric pencil $\lambda A(t) - B(t)$ is also invariant.*

Proof. By the Lemma 3.1, the roots in λ of the characteristic equation $f(\lambda; t) = \det(\lambda A(t) - B(t))$ can be written as four continuous complex-valued functions $\lambda_i(t),\ i = 1, 2, 3, 4$. The subscript k of the Segre characteristic defines the number of $\lambda_i(t)$ that are real-valued within the time span, $k \in \{0, 1, 2, 3, 4\}$. We suppose $\hat{k}$ of these real-valued $\lambda_i(t)$ are distinct, and furthermore, throughout the given time span the order of these λ_i does not flip otherwise there must be a time instant t_0 when two of the $\lambda_i(t_0)$ cross each other and this crossing leads to a change of the Segre characteristic. So we can reorder these functions as $\lambda_1(t) < \cdots < \lambda_{\hat{k}}(t)$, $\hat{k} \leq k$, which partition the λ-axis into $\hat{k}+1$ intervals, denoted by $\mathcal{I}_i = (\lambda_i(t), \lambda_{i+1}(t)), i = 0, \cdots, \hat{k}$, where $\lambda_0(t) := -\infty, \lambda_{\hat{k}+1}(t) := +\infty$.

We shall first prove that the numbers in the index sequence of $(\mathcal{A}(t), \mathcal{B}(t))$ are invariant. Suppose that the number in the j-th position of the index sequence, i.e. representing the index value $\mathrm{Id}(\lambda^*(t), t)$, where $\lambda^*(t) \in \mathcal{I}_j(t)$ is a continuous real-valued function, changes during the given time span. The eigenvalues of the matrix $\lambda^*(t)A(t) - B(t)$ are real roots of the function $\det(\lambda^*(t)A(t) - B(t) - uI) = 0$ in u, which can be written as four continuous real-valued functions $u_i(t), i = 1, 2, 3, 4$. Now since $\mathrm{Id}(\lambda^*(t), t)$ changes, at least one u_i changes sign, say $u_1(t)$. So there exists an instant t_0 when $u_1(t_0) = 0$, which when substituted back yields $\det(\lambda^*(t_0)A(t_0) - B(t_0)) = 0$. This zero implies that $\lambda^*(t_0)$ is a root of the characteristic function $f(\lambda; t_0) = 0$, which contradicts the fact that $\lambda^*(t) \in \mathcal{I}_k(t)$. Therefore, the numbers in the index sequence of $(\mathcal{A}(t), \mathcal{B}(t))$ are invariant.

By the tables given in [Tu et al. (2009)], once the Segre characteristic is fixed, the numbers in the index sequence can directly derive the index sequence, except for the case when number 2 appears in the Segre characteristic, which might correspond to the Jordan blocks $\wr\wr_+$ or $\wr\wr_-$ in the index sequence. We next prove that the sign of such a Jordan block is also invariant during the given time span. Suppose that $\lambda_0(t)$ is a double root of $f(\lambda; t) = 0$ whose Jordan block is of order 2. Also suppose that this order 2 Jordan block has sign $+$ at time t_1, but has sign $-$ at time t_2. Then the eigencurve of $\det(\lambda A(t_1) - B(t_1) - uI) = 0$ is quadratic in a small neighborhood of $\lambda_0(t_1)$, with $u_1(\lambda_0) = 0$ and $u_2(\lambda_0) < 0$. Similarly, the eigencurve of $\det(\lambda A(t_2) - B(t_2) - uI) = 0$ is quadratic in a small neighborhood of $\lambda_0(t_2)$, with $u_1(\lambda_0) > 0$ and $u_2(\lambda_0) = 0$. Since $u_i(\lambda_0(t), t), i = 1, 2$ are both continuous functions in t, there must be an instant $t_1 < t_0 < t_2$ such that $u_1(t_0) > 0, u_2(t_0) < 0$ or $u_1(t_0) = u_2(t_0) = 0$. The former case indicates that $\lambda_0(t_0)$ is no longer a root of $f(\lambda; t_0)$, while the latter case indicates that the Jordan block associated with $\lambda_0(t_0)$ becomes $[(11)]$. Both of these two results contradict the fact that the Segre characteristic is invariant during the given time span. Therefore the sign of such a Jordan block is also invariant during the given time span.

Thus we conclude that the index sequence is invariant during the time span. □

3.2. Detection of the Segre characteristic change

From the above observations, Step 1 in the outline of our approach is reduced to

1. detecting the instants $t_i, i = 1, \cdots, n$ across which the Segre characteristic of the quadric pencil $\lambda A(t) - B(t)$ changes, which contains the following two levels:
   1.1 Detect the instants t_i when the root pattern in λ of the function $f(\lambda; t) = 0$ changes;
   1.2 Detect the instants t_i when the Segre characteristic of $\lambda A(t) - B(t)$ changes.

Note 1. *By the root pattern we include information on the number of real roots, the number of complex roots, as well as the multiplicity of each root. Note that the change of the root pattern in λ of the function $f(\lambda; t) = 0$ necessarily leads to a change of the Segre characteristic of $\lambda A(t) - B(t)$, but the converse is not true, because the Segre characteristics essentially reveals a lot finer information about the roots of $f(\lambda; t) = 0$ in λ, which are, the Jordan blocks associated with these roots in λ in the Jordan normal form of the matrix $A^{-1}(t)B(t)$.*

If $f(\lambda; t)$ has a multiple factor of positive degree in λ, i.e., $f(\lambda; t) = g(\lambda; t)^m \tilde{f}(\lambda; t)$ for some $m > 1$, $\deg_\lambda(g) \geq 1$ and $\gcd(g, \tilde{f}) = 1$, the moving quadric pencil $\lambda A(t) - B(t)$ is said to be a *non-generic pencil*;

otherwise the moving quadric pencil is *generic*. In the following we shall separately treat these two different cases of $f(\lambda; t)$.

3.2.1. Generic Pencils

We first consider generic moving quadric pencils of $\mathcal{A}, \mathcal{B}$, i.e., $f(\lambda; t)$ does not have any multiple factor which has a positive degree in λ. Hence clearly, for a generic t the characteristic function $f(\lambda; t) = 0$ has four different simple roots. Step 1.1 is completed by the following theorem:

Theorem 3.3. *Suppose that the moving quadric pencil $\lambda A(t) - B(t)$ is generic. The root pattern in λ of $f(\lambda; t) = 0$ changes only at the instants t_i that are real roots of $\mathrm{Res}_\lambda(f, f_\lambda)(t) = 0$.*

Proof. $\mathrm{Res}_\lambda(f, f_\lambda) = 0$ gives all the instants t_i when $f(\lambda; t_i) = 0$ has a multiple root in λ. Except at these instants t_i the function $f(\lambda; t) = 0$ always has four simple roots, all real, or all complex, or two real and two complex. These three cases can not switch to another without passing through an instant t_i that satisfies $\mathrm{Res}_\lambda(f, f_\lambda)(t) = 0$. $\square$

The following theorem indicates that Step 1.2 is unnecessary for generic moving quadric pencils.

Theorem 3.4. *Suppose that the moving quadric pencil $\lambda A(t) - B(t)$ is generic. Let $t_i, i = 1, \cdots, n$ be all the different real roots of $\mathrm{Res}_\lambda(f, f_\lambda)(t) = 0$, and let $t_0 = -\infty, t_{n+1} = +\infty$. The Segre characteristic of this moving quadric pencil is invariant during each interval $t_i < t < t_{i+1}, i = 0, \cdots, m$.*

Proof. By Theorem 3.3, during each interval $t_i < t < t_{i+1}, i = 0, \cdots, m$, the root pattern in λ of $f(\lambda; t) = 0$ does not change, which must be one of the three cases: all real, or all complex, or two real and two complex. Each of these three root patterns corresponds to only one Segre characteristic from $[1111]_0$, $[1111]_2$ and $[1111]_4$. Therefore, the Segre characteristic of this moving quadric pencil is also invariant during each interval. $\square$

By Theorem 3.2 we directly have

Corollary 3.1. *Suppose that the moving quadric pencil $\lambda A(t) - B(t)$ is generic. Let $t_i, i = 0, \cdots, n+1$ be those instants in Theorem 3.4. Then the QSIC type does not change during each interval $t_i < t < t_{i+1}, i = 0, \cdots, n$.*

Example 3.1. *Given a hyperbolic paraboloid $\mathcal{A} : X^T A X = 0$ and a rotating circular paraboloid $\mathcal{B} : X^T B(t) X = 0$ with $B(t) = T_B^T \hat{B} T_B$, where*

$$A := \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & & -1 \\ & & -1 & \end{pmatrix}, \quad \hat{B} := \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & & -1 \\ & & -1 & \end{pmatrix}, \quad T_B = \begin{pmatrix} 1 & & & \\ & \frac{2t}{1+t^2} & -\frac{1-t^2}{1+t^2} & \\ & \frac{1-t^2}{1+t^2} & \frac{2t}{1+t^2} & \\ & & & 1 \end{pmatrix}.$$

The normalized characteristic function associated with $\mathcal{A}$ and $\mathcal{B}$ is

$$f(\lambda;t) = (\lambda-1)(1+\lambda)(\lambda^2+2\lambda^2t^2+\lambda^2t^4+2\lambda t^2-4\lambda t^3-4t\lambda-\lambda-\lambda t^4+1+2t^2+t^4),$$

hence the moving quadric pencil is generic. Compute

$$\begin{aligned}\mathrm{Res}_\lambda(f,f_\lambda) = {} & (-8t^3-8t+12t^2+2+2t^4)(-8t^3-8t-4t^2-6-6t^4)(9t^{20}-48t^{19}+106t^{18}-112t^{17} \\ & -11t^{16}+256t^{15}-456t^{14}+384t^{13}+2t^{12}-480t^{11}+700t^{10}-480t^9+2t^8+384t^7-456t^6 \\ & +256t^5-11t^4-112t^3+106t^2-48t+9),\end{aligned}$$

which has two real roots $t_1=-1, t_2=1$. By Theorem 3.4 the QSIC type can only change at $t=t_1$ and $t=t_2$. Now we choose $s_1=-2<t_1<s_2=0<t_2<s_3=2$. By direct computation we get that the index sequences of the matrix $\lambda A(s_i)-B(s_i)$ are all $\langle 1|2|3\rangle$ for $i=1,2,3$, hence by the table given in [Tu et al. (2009)] the intersection curve of the two quadrics is a simple real loop. The index sequence of the matrix $\lambda A(t_i)-B(t_i)$ are both $\langle 1|||2|3\rangle$ for $i=1,2$, which implies that the intersection curve of the two quadrics is a double real loop. The whole process of the QSIC change is shown in Figure 1.

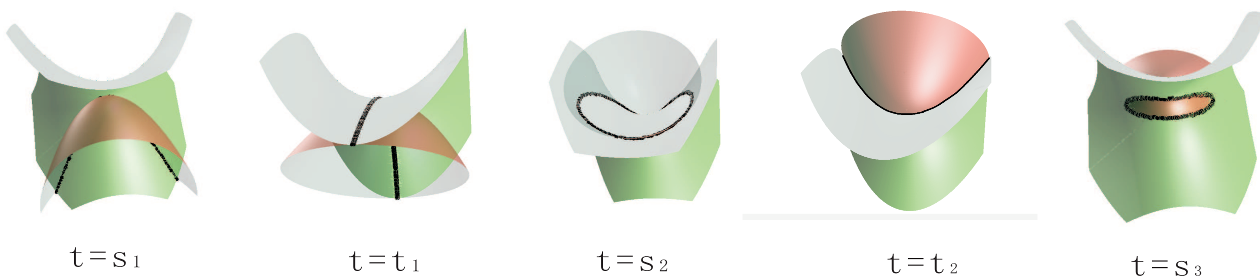

Figure 1: The change of the QSIC for a generic moving quadric pencil.

3.3. Non-generic Pencils

Next we discuss how to detect the Segre characteristic changes for the non-degenerate moving quadric pencils, where $f(\lambda;t)$ can be factorized into

$$f(\lambda;t) = g^m(\lambda;t)\tilde{f}(\lambda;t), \text{ with } \deg_\lambda(g)\geq 1, \gcd(g,\tilde{f})=1, m\in\{2,3,4\},$$

and $g(\lambda;t)$ is irreducible in $\mathbb{R}(t)[\lambda]$. We shall treat the two cases of $\deg_\lambda(g)=1$ and $\deg_\lambda(g)=2$ separately. Throughout this section we shall use the notation $M(t)=A(t)^{-1}B(t)$.

3.3.1. $\deg_\lambda(g) = 1$

We shall discuss the three subcases $m = 2, 3, 4$ separately.

• $\mathbf{m = 2}$:

Since $\deg_\lambda(\tilde{f}) = 2$, there could be three different situations: $\tilde{f}$ is irreducible, or $\tilde{f} = h_1(\lambda; t)h_2(\lambda; t)$ with $h_1 \neq h_2$ both of degree one in λ; or $\tilde{f} = h^2(\lambda; t)$. Step 1.1 is given by $\{t|\mathrm{Res}_\lambda(g, \tilde{f}) = 0 \text{ or } \mathrm{Res}_\lambda(\tilde{f}, \tilde{f}_\lambda) = 0\}$ for the first and second case, and $\{t|\mathrm{Res}_\lambda(g, h) = 0\}$ for the third case. The following are for step 1.2.

Theorem 3.5. *Suppose that $f(\lambda; t) = g^2(\lambda; t)\tilde{f}(\lambda; t)$, where $\deg_\lambda(g) = 1$ and $\tilde{f}$ is irreducible. For a generic t_0 such that $g(\lambda; t_0)$ has no common factor with $\tilde{f}(\lambda; t_0)$ and $\tilde{f}(\lambda; t_0)$ has no double root, the Segre characteristic of the quadric pencil $\lambda A(t_0) - B(t_0)$ is*

- $[211]_r \Leftrightarrow g\tilde{f}(M(t_0); t_0) \neq \mathbf{0}$*;*
- $[(11)11]_r \Leftrightarrow g\tilde{f}(M(t_0); t_0) = \mathbf{0}$*,*

where $r = 1$ or 3.

Proof. Since $\deg_\lambda(\tilde{f}) = 2$, $\tilde{f}(\lambda; t_0)$ has two roots λ_1, λ_2, both real or both complex, which by assumption are different to each other. Hence considering $g^2(\lambda; t_0)$ has a double real root λ_0, we have $r = 1$ or 3.

"$\Rightarrow$":

If the Segre characteristic is $[211]_r$, by Lemma 2.1 the minimal polynomial of the Jordan block associated with λ_0 is $(\lambda - \lambda_0)^2$, and the minimal polynomial of the Jordan block associated with λ_1, λ_2 is $(\lambda - \lambda_1)(\lambda - \lambda_2) = \tilde{f}(\lambda; t_0)$. So by Lemma 2.2 the minimal polynomial of the matrix $M(t_0)$ is $(\lambda - \lambda_0)^2\tilde{f}(\lambda; t_0) = f(\lambda; t_0)$. Hence $g\tilde{f}(M(t_0); t_0) \neq \mathbf{0}$.

If the Segre characteristic is $[(11)11]_r$, by Lemma 2.1 the minimal polynomial of the Jordan block associated with λ_0 is $\lambda - \lambda_0$, and the minimal polynomial of the Jordan block associated with λ_1, λ_2 is $(\lambda - \lambda_1)(\lambda - \lambda_2) = \tilde{f}(\lambda; t_0)$. Hence the minimal polynomial of the matrix $M(t_0)$ is $(\lambda - \lambda_0)\tilde{f} = g\tilde{f}(\lambda; t_0)$. So $g\tilde{f}(M(t_0); t_0) = \mathbf{0}$.

"$\Leftarrow$": This direction follows in a straightforward manner from the conclusion in the other direction. □

Theorem 3.6. *Suppose that $f(\lambda; t) = g^2(\lambda; t)h_1(\lambda; t)h_2(\lambda; t)$, where $\deg_\lambda(g) = 1$ and $\tilde{f}$ is irreducible. For a generic t_0 such that $g(\lambda; t_0)$ has no common factor with $h_1h_2(\lambda; t_0)$ and $h_1(\lambda; t_0)$ has no common root with $h_2(\lambda; t_0)$, the Segre characteristic of the quadric pencil $\lambda A(t_0) - B(t_0)$ is*

- $[211]_3 \Leftrightarrow gh_1h_2(M(t_0); t_0) \neq \mathbf{0}$*;*
- $[(11)11]_3 \Leftrightarrow gh_1h_2(M(t_0); t_0) = \mathbf{0}$*.*

Proof. Notice that the two roots of $\tilde{f}(\lambda; t_0) = h_1(\lambda; t_0)h_2(\lambda; t_0)$ are now both real, hence the subscript in the Segre characteristic is now 3. The rest of the proof is exactly the same as Theorem 3.5. □

Theorem 3.7. *Suppose that $f(\lambda;t) = g^2(\lambda;t)h^2(\lambda;t)$, where $\deg_\lambda(g) = \deg_\lambda(h) = 1$. For a generic t_0 such that $g(\lambda;t_0)$ has no common factor with $h(\lambda;t_0)$, the Segre characteristic of the quadric pencil $\lambda A(t_0) - B(t_0)$ is*

- $[22]_2 \Leftrightarrow gh^2(M(t_0);t_0) \neq \mathbf{0}$ *and* $g^2h(M(t_0);t_0) \neq \mathbf{0}$;
- $[2(11)]_2 \Leftrightarrow gh^2(M(t_0);t_0) = \mathbf{0}$ *or* $g^2h(M(t_0);t_0) = \mathbf{0}$, *but* $gh(M(t_0);t_0) \neq \mathbf{0}$;
- $[(11)(11)]_2 \Leftrightarrow gh(M(t_0);t_0) = \mathbf{0}$.

Proof. The proof is similar to the proof of Theorem 3.5 by using Lemma 2.1 and Lemma 2.2. □

• $\mathbf{m = 3}$:

Step 1.1 is given by $\{t|\mathrm{Res}_\lambda(g,\tilde{f}) = 0\}$. The following are for step 1.2.

Theorem 3.8. *Suppose that $f(\lambda;t) = g^3(\lambda;t)\tilde{f}(\lambda;t)$, where $\deg_\lambda(g) = \deg_\lambda(\tilde{f}) = 1$. For a generic t_0 such that $g(\lambda;t_0)$ has no common factor with $\tilde{f}(\lambda;t_0)$, the Segre characteristic of the quadric pencil $\lambda A(t_0) - B(t_0)$ is*

- $[31]_2 \Leftrightarrow g^2\tilde{f}(M(t_0);t_0) \neq \mathbf{0}$;
- $[(21)1]_2 \Leftrightarrow g^2\tilde{f}(M(t_0);t_0) = \mathbf{0}, g\tilde{f}(M(t_0);t_0) \neq \mathbf{0}$;
- $[(111)1]_2 \Leftrightarrow g\tilde{f}(M(t_0);t_0) = \mathbf{0}$.

Proof. The proof is similar to the proof of Theorem 3.5 by using Lemma 2.1 and Lemma 2.2. □

• $\mathbf{m = 4}$:

Step 1.1 is unnecessary since for each t the function $f(\lambda;t) = 0$ always has a unique root of multiplicity four in λ. We go directly to step 1.2.

Theorem 3.9. *Suppose that $f(\lambda;t) = g^4(\lambda;t)$, where $\deg_\lambda(g) = 1$. The Segre characteristic of the quadric pencil $\lambda A(t_0) - B(t_0)$ is*

- $[4]_1 \Leftrightarrow g^3(M(t_0);t_0) \neq \mathbf{0}$
- $[(31)]_1 \Leftrightarrow g^3(M(t_0);t_0) = \mathbf{0}, g^2(M(t_0);t_0) \neq \mathbf{0}$
- $[(22)]_1$ *or* $[(211)]_1 \Leftrightarrow g^2(M(t_0);t_0) = \mathbf{0}, g(M(t_0);t_0) \neq \mathbf{0}$
- $[(1111)]_1 \Leftrightarrow g(M(t_0);t_0) = \mathbf{0}$.

Proof. The proof is similar to the proof of Theorem 3.5 by using Lemma 2.1 and Lemma 2.2. □

The following theorem is to distinguish $[(22)]_1$ from $[(211)]_1$ in the third bullet in Theorem 3.9:

Theorem 3.10. *Suppose that* $f(\lambda;t) = g^4(\lambda;t)$*, where* $\deg_\lambda(g) = 1$*. Solve* $\lambda = \lambda(t)$ *from* $g(\lambda;t) = 0$ *(since* $\deg_\lambda(g) = 1$*). Then the Segre characteristic of the quadric pencil* $\lambda A(t_0) - B(t_0)$ *is*

- $[(22)]_1 \Leftrightarrow g^2(M(t_0);t_0) = \mathbf{0}, g(M(t_0);t_0) \neq \mathbf{0}$ *and* $\mathrm{rank}(M(t_0) - \lambda(t_0)I) = 2$*;*
- $[(211)]_1 \Leftrightarrow g^2(M(t_0);t_0) = \mathbf{0}, g(M(t_0);t_0) \neq \mathbf{0}$ *and* $\mathrm{rank}(M(t_0) - \lambda(t_0)I) = 1$*.*

Proof. Since $f(\lambda;t) = g^4(\lambda;t)\tilde{f}(t)$, the root of $g(\lambda;t_0)$ is the eigenvalue (of multiplicity 4) of the matrix $M(t_0)$. Suppose that $M(t_0) = P^{-1}JP$, where J is the Jordan normal form of the matrix $M(t_0)$ and P an invertible matrix. Solving $\lambda = \lambda(t)$ from $g(\lambda;t) = 0$, we have $\mathrm{rank}(M(t_0) - \lambda(t_0)I) = \mathrm{rank}(P^{-1}JP - P^{-1}\lambda(t_0)IP) = \mathrm{rank}(J - \lambda(t_0)I)$, which is equal to 2 if J is of the form $[(22)]_1$ and 1 if J is of the form $[(211)]_1$. □

Note 2. *Since* $\deg_\lambda(g) = 1$*,* $\lambda = \lambda(t)$ *can directly be solved from* $g(\lambda;t) = 0$*. If the motions in* t *are given by polynomials, we can just perform Gaussian elimination to the polynomial matrix* $N(t) := M(t) - \lambda(t)I$ *and solve for those instants* t_0 *for which* $\mathrm{rank}(N(t_0)) = k$*, where* $k = 1$ *or 2. Otherwise if the motions in* t *are not represented by polynomials, we can step back to use the definition of the rank of a matrix, i.e., the largest size of the non-zero minor of the matrix, to solve for the common zeros of all the* $k+1$ *minors of the matrix* $N(t)$*, which are those instants* t_0 *such that* $\mathrm{rank}(N(t_0)) = k$*.*

From the above observations, we derive the following algorithm of detecting the Segre characteristic variations of the quadric pencil $\lambda A(t) - B(t)$ for the degenerate case $\deg_\lambda(g) = 1$:

3.3.2. $\deg_\lambda(g) = 2$

Step 1.1 can be given by $\{t|\mathrm{Res}_\lambda(g, g_\lambda) = 0\}$. We next consider step 1.2.

Theorem 3.11. *Suppose that* $f(\lambda;t) = g^2(\lambda;t)$*, where* $g(\lambda;t)$ *is irreducible over* $\mathbb{R}[\lambda;t]$ *and* $\deg_\lambda(g) = 2$*. For a generic* t_0 *such that* $g(\lambda;t_0)$ *has two different simple roots, the Segre characteristic of the quadric pencil* $\lambda A(t_0) - B(t_0)$ *is*

- $[22]_r$ *or* $[2(11)]_2 \Leftrightarrow g(M(t_0);t_0) \neq \mathbf{0}$
- $[(11)(11)]_r \Leftrightarrow g(M(t_0);t_0) = \mathbf{0}$*,*

where $r = 0$ *or 2.*

Input : The normalized characteristic polynomial $f(\lambda;t) = g^m \tilde{f}$ of two quadric forms $A(t)$ and $B(t)$, where $\deg_\lambda(g) = 1$.
Output: Candidate instants $T = \{t_i\}$ when the Segre characteristic changes.

```
begin
    T ← ∅; M(t) := A^{-1}(t)B(t);
    foreach i = 1, ··· , m − 1 do
        Let f_i = f/g^i;
        Substitute λ = M(t) to f_i into get matrix M_i(t);
        if M_i(t) ≢ 0 then T ← {t|M_i(t) = 0};                      // Theorem 3.5--3.9
    end
    if m = 2 then
        if f̃ = h^2 then
            T ← {t|Res_λ(g, h)} = 0;
            let f_2 := f/h and f_3 := f/gh;
            for i = 2, 3 do
                Substitute λ = M(t) into f_i to get the matrix M_i(t);
                if M_i(t) ≢ 0 then
                    T ← {t|M_i(t) = 0}                                  // Theorem 3.7
                end
            end
        end
        else if f̃ ≠ h^2 then
            T ← {t|Res_λ(f̃, f̃_λ)} = 0;  T ← {t|Res_λ(g, f̃)} = 0;
        end
    end
    if m = 3 then
        T ← {t|Res_λ(g, f̃)} = 0;
    end
    if m = 4 and {t|M_2(t) = 0} ≠ ∅ then
        Solve λ = λ(t) from g(λ; t) = 0                                 // Theorem 3.10;
        Substitute λ = λ(t) into matrix M(t) − λI to get the matrix N(t);
        if rank(N(t)) = 3 for generic t then
            T ← {t|common roots of all order 3 minors of N(t)} = 0;
            T ← {t|common roots of all order 2 minors of N(t)} = 0;
        end
        else if rank(N(t)) = 2 for generic t then
            T ← {t|common roots of all order 2 minors of N(t)} = 0
        end
    end
end
```

Algorithm 1: Detecting instants for the change of the Segre characteristic of a non-generic moving quadric pencil with $\deg_\lambda(g) = 1$.

Proof. Since $\deg_\lambda(g) = 2$, by assumption $g(\lambda; t_0)$ has two roots $\lambda_1 \neq \lambda_2$, both real or both complex, so $r = 2$ or 0. If the Segre characteristic is $[(11)(11)]_r$, by Lemma 2.1 and Lemma 2.2 the minimal polynomial of the matrix $M(t_0)$ is $(\lambda - \lambda_1)(\lambda - \lambda_2) = g(\lambda; t_0)$; if the Segre characteristic is $[22]_r$, the minimal polynomial of the matrix $M(t_0)$ is $(\lambda - \lambda_1)^2(\lambda - \lambda_2)^2 = g^2(\lambda; t_0)$; if the Segre characteristic is $[2(11)]_2$, the minimal polynomial of the matrix $M(t_0)$ is $(\lambda - \lambda_1)^2(\lambda - \lambda_2)$ or $(\lambda - \lambda_1)(\lambda - \lambda_2)^2$, depending on whether the 2×2 Jordan block belongs to λ_1 or λ_2. These observations directly yield the conclusion. □

The next theorem helps to distinguish $[22]_{r=0 \text{ or } 2}$ from $[2(11)]_2$ in the previous theorem.

Theorem 3.12. *Under all the assumptions in Theorem 3.11, solve for λ from $g(\lambda; t)$ to get $\lambda_1(t), \lambda_2(t)$ (a square root is involved in here). For a parameter t_0, suppose that $g(\lambda; t_0)$ has two different simple roots. Then the Segre characteristic of the two quadrics at t_0 is*

- $[22]_{r=0 \ or \ 2} \Leftrightarrow g(M(t_0); t_0) \neq \mathbf{0}, \mathrm{rank}(M(t_0) - \lambda_i(t_0)I) = 3$ *for* $i = 1, 2$
- $[2(11)]_2 \Leftrightarrow g(M(t_0); t_0) \neq \mathbf{0}, \mathrm{rank}(M(t_0) - \lambda_i(t_0)I) = 2$ *for* $i = 1$ *or 2,*

Proof. Suppose that $M(t_0) = P^{-1}JP$, where J is the Jordan form of the matrix $M(t_0)$, and P is an invertible matrix. If the Segre characteristic is $[22]_2$,

$$\begin{aligned} & M(t_0) - \lambda_1(t_0)I = P^{-1}JP - \lambda_1(t_0)I \\ &= \begin{pmatrix} \lambda_1 & 1 & & \\ & \lambda_1 & & \\ & & \lambda_2 & 1 \\ & & & \lambda_2 \end{pmatrix} - \lambda_1 I = \begin{pmatrix} 0 & 1 & & \\ & 0 & & \\ & & \lambda_2 - \lambda_1 & 1 \\ & & & \lambda_2 - \lambda_1 \end{pmatrix}, \end{aligned}$$

hence $\mathrm{rank}(M(t_0) - \lambda_1 I) = 3$; similarly $\mathrm{rank}(M(t_0) - \lambda_2 I) = 3$. If the Segre characteristic is $[22]_0$, suppose the two complex roots of $g(\lambda; t_0)$ are $a \pm bi$. Then

$$\begin{aligned} & M(t_0) - \lambda_1(t_0)I = P^{-1}JP - \lambda_1(t_0)I \\ &= \begin{pmatrix} a & -b & 1 & \\ b & a & & 1 \\ & & a & -b \\ & & -b & a \end{pmatrix} - (a + bi)I = \begin{pmatrix} -bi & -b & 1 & \\ & -bi & & 1 \\ & & -bi & -b \\ & & & -bi \end{pmatrix}, \end{aligned}$$

whose rank is 3, similarly $\mathrm{rank}(M(t_0) - \lambda_2 I) = 3$.

If the Segre characteristic is $[2(11)]_2$, the polynomial $f(\lambda; t_0)$ has two real roots. If the 2×2 Jordan block is associated to λ_1, then $\mathrm{rank}(M(t_0) - \lambda_1 I) = 3$ and $\mathrm{rank}(M(t_0) - \lambda_2 I) = 2$; if the 2×2 Jordan block is associated to λ_2, then $\mathrm{rank}(M(t_0) - \lambda_1 I) = 2$ and $\mathrm{rank}(M(t_0) - \lambda_2 I) = 3$. □

Note 3. *Suppose that* $g(\lambda;t) = g_2(t)\lambda^2 + g_1(t)\lambda + g_0(t)$. *Solving for* λ *from* $g(\lambda;t)$ *yields*

$$\lambda_{1,2}(t) = \frac{-g_1 \pm \sqrt{g_1^2 - 4g_0g_2}}{2g_2}.$$

We substitute this representation into the matrix $N(t) := M(t) - \lambda_i(t)I$. *To solve for the instants* t_0 *for which* $\mathrm{rank}(N(t_0)) = k$, *we just solve for the common roots of all the size* $k+1$ *minors of the matrix* $N(t)$. *Note that in these minors square roots are involved.*

Input : The characteristic polynomial $f(\lambda;t) = g^2(\lambda;t)$ of two quadric forms $A(t)$ and $B(t)$, $\deg_\lambda(g) = 2$, g irreducible
Output: Candidate instants $T = \{t_i\}$ when the Segre characteristic changes.
begin
  $T \leftarrow \emptyset$; $M(t) := A^{-1}(t)B(t)$;
  Substitute $\lambda = M(t)$ into $g(\lambda;t)$ to get the matrix $M_1(t)$;
  if $M_1(t) \not\equiv \mathbf{0}$ **then**
    $T \leftarrow \{t | M_1(t) = \mathbf{0}\}$
  end
  Solve $\lambda = \lambda_{1,2}(t)$ from $g(\lambda;t) = 0$;
  for $i = 1, 2$ **do**
    Substitute $\lambda = \lambda_i(t)$ into matrix $M(t) - \lambda I$ to get the matrix $N_i(t)$;
  end
  if $\mathrm{rank}(N(t)) = 3$ *for generic* t **then**
    $T \leftarrow \{t|\text{common roots of all order 3 minors of } N(t)\} = 0$;
  end
end

Algorithm 2: Detecting instants for the change of the Segre characteristic of a non-generic moving quadric pencil with $\deg_\lambda(g) = 2$.

4. Algorithm and Examples

We now combine the generic and non-generic cases together in Algorithm 3. Examples are then provided to illustrate the algorithm.

Example 4.1. *Given two moving quadrics* $\mathcal{A} : X^T T_A^T A T_A X = 0$ *and* $\mathcal{B} : X^T T_B^T B T_B X = 0$, *where*

$$A = \mathrm{diag}(1, 1, 1, -1), \qquad B = \mathrm{diag}(\frac{1}{(2-\frac{3}{2}t)^2}, 4, 4, -1),$$

and

$$T_A = \begin{pmatrix} 1 & & & 1 \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \quad T_B = \begin{pmatrix} 1 & & & 2-\frac{3}{2}t \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

Input : The characteristic polynomial $\hat{f}(\lambda;t)$ of two quadric forms $A(t)$ and $B(t)$, the leading coefficient $l(t)$ of $\hat{f}(\lambda;t)$ in λ;
Output: Time interval $\mathcal{I}_i$ and the topology of the QSIC during $\mathcal{I}_i$; instants t_i and the topology of the QSIC at $t=t_i$.
begin
  $\{T \leftarrow \{t | l(t)=0$ or $l(t)=\infty\}$;
  let $f(\lambda;t)=\hat{f}(\lambda;t)/l(t)$; factor $f(\lambda;t)$;
  if $f(\lambda;t)$ *has no multiple factor* **then**
    $T \leftarrow \{t|\mathrm{Res}_\lambda(f,f_\lambda)(t)=0\}$
  end
  else if $f=g^m(\lambda;t)\tilde{f}(\lambda;t)$, *where* $\deg_\lambda(g)=1$ **then**
    Use Algorithm 1 to compute T;
  end
  else if $f=g^2(\lambda;t)$ *where* $\deg_\lambda(g)=2$ *and g is irreducible* **then**
    Use Algorithm 2 to compute T;
  end
  Reorder elements in T as $t_1<t_2<\cdots<t_n$; Set $t_0=-\infty$ and $t_{n+1}=+\infty$;
  Let $\mathcal{I}_i=(t_{i-1},t_i), i=1,\cdots,n+1$;
  Select real number $s_i\in\mathcal{I}_i, i=1,\cdots,n+1$;
  Use Algorithm 1 in Tu et al. (2009) to compute the QSIC at $t=s_i$;
  Use Algorithm 1 in Tu et al. (2009) to compute the QSIC at $t=t_i$;
end

Algorithm 3: Computing the QSIC variations of two moving quadrics.

The characteristic equation associated with $\mathcal{A}$ and $\mathcal{B}$ is

$$f(\lambda;t)=\frac{(4\lambda+3\lambda t+2)^2(\lambda-4)^2}{(-4+3t)^2}.$$

The denominator of f vanishes when $t=\frac{4}{3}$, because the quadric $\mathcal{B}$ degenerates to a double plane. Hence we treat $t=\frac{4}{3}$ as a special instant in time. Besides this, observe that for a generic $t\in(-\infty,+\infty)$, $f(\lambda;t)=0$ always has two different double real roots, except for $t=\frac{7}{6}$ when $f(\lambda;t)=0$ has a root of multiplicity four. Further checking the variation of the Segre characteristic, we find that the Segre sequence is $[2(11)]_2$ for a generic $t\in(-\infty,+\infty)$, except at instants $t=\frac{2}{3}$ and $t=\frac{7}{6}$. When $t=\frac{2}{3}$, the Segre characteristic is $[(11)(11)]_2$; when $t=\frac{7}{6}$, the Segre characteristic is $[(211)]_1$.

Let $t_1=\frac{2}{3}, t_2=\frac{7}{6}, t_3=\frac{4}{3}$. For $i=1,2$, the index sequence of the quadric pencil $\lambda A(t_i)-B(t_i)$ is $\langle 1\|1\|3\rangle$, $\langle 1 \wr \wr_- \|3\rangle$, respectively, corresponding to case 30 and case 33 of Table 3 in [Tu et al. (2009)]. At $t=t_3$, the quadric $\mathcal{B}$ degenerates to a double plane $x=0$, which intersects with the quadric $\mathcal{A}$ at the point $(0,0,0,1)$.

Now $t_i, i=1,2,3$ partition the time axis into four intervals $\mathcal{I}_i$. For each interval we choose an arbitrary representative, e.g., $s_1=\frac{1}{2}<t_1<s_2=1<t_2<s_3=\frac{5}{4}<t_3<s_3=\frac{3}{2}$. The Segre characteristic of the quadric pencil $\lambda A(s_i)-B(s_i)$ is $\langle 1\wr\wr_-1\|3\rangle$, $\langle 1\wr\wr_+1\|3\rangle$, $\langle 1\wr\wr_-1\|3\rangle$, $\langle 1\wr\wr_+1\|3\rangle$, respectively, sequentially corresponding to case 24, case 25, case 24, case 25 of Table 3 in [Tu et al. (2009)]. The type of the intersection curve $\mathcal{C}$ of the two moving quadrics is invariant during each interval $\mathcal{I}_i$, which is given by the

results from $t = s_i$, $i = 1, 2, 3, 4$.

The whole process of the QSIC change is shown in Figure 2. In the following theory we shall explain how to detect the instants $t_i, i = 1, 2, 3$ that are crucial to the variation of the QSIC.

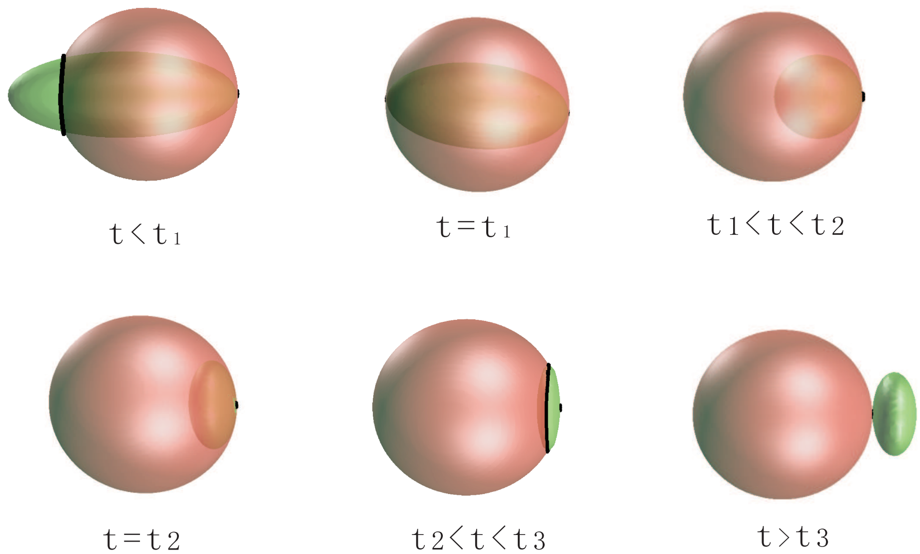

Figure 2: The change of QSIC of two moving ellipsoids

Example 4.2. *Given a hyperboloid of one sheet $\mathcal{A} : X^t AX = 0$ and a moving sphere $\mathcal{B} : X^t T^t BTX = 0$, where*

$$A := \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}, \quad B := \begin{pmatrix} \frac{1}{2} & & & \\ & \frac{1}{2} & & \\ & & \frac{1}{2} & \\ & & & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & t \\ & & & 1 \end{pmatrix}.$$

The characteristic polynomial is

$$f(\lambda; t) = (2\lambda - 1)^2(2\lambda^2 - \lambda + \lambda t^2 - 1).$$

Let $g = 2\lambda - 1, \tilde{f} = 2\lambda^2 - \lambda + \lambda t^2 - 1$. Substitute $\lambda = A^{-1}T^t BT$ into $g\tilde{f}$ we get a zero matrix, which means that except for a finite number of t_0 for which $g(\lambda)$ and $\tilde{f}(\lambda; t_0)$ has a common root, at all generic instants t the Segre characteristic of the matrix $\lambda = A^{-1}T^t BT(t)$ is $[(11)11]_r$. Now compute

$$\mathrm{Res}_\lambda(g(\lambda; \tilde{f}(\lambda; t))) = 2(t^2 - 2).$$

Hence $t_1 = -\sqrt{2}$ and $t_2 = \sqrt{2}$ are the instants when the QSIC type changes. Actually when $t < t_2$ or $t > t_2$, the index sequences are both $\langle 0|1|2|3|4\rangle$, so the QSIC is an imaginary loop. When $t_1 < t < t_2$, the index *sequence is $\langle 1|2|1|2|3\rangle$, so the QSIC is two real loops. When $t = t_1$ or $t = t_2$, the index sequence is $\langle 1|||2|3\rangle$, so the QSIC is a real loop. See Figure 3 for illustration.*

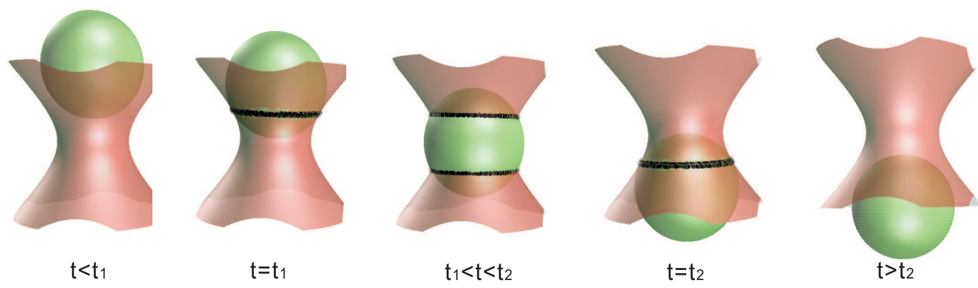

Figure 3:

5. Conclusion and Future Work

We provide a novel algebraic approach for detecting the variation of the types of the intersection curve of two moving quadrics in $\mathbb{PR}^3$, real projective 3-space. The key step is to compute the discrete instants across which the type of the QSIC changes. Once these discrete instants are found, [Tu et al. (2009)] can help us to decide the type of the QSIC during each of the intervals partitioned by these discrete instants. The core mathematical underpinnings of our algorithm requires detecting the variations of the Jordan form of the matrix $M(t) = A^{-1}(t)B(t)$, which essentially reduces to observing the changes of the minimal polynomial of the Jordan form.

We explore possible future work by analyzing Example 3.1. We detected two instants t_1, t_2, at which the QSIC is a double real loop. Except at t_1, t_2 the QSIC is always a simple real loop. However, since we are studying the QSIC in $\mathbb{PR}^3$, we cannot further detect those instants when the QSIC changes from an open curve to a closed loop, or conversely from a closed loop to an open loop in $\mathbb{R}^3$, the real affine space. Therefore, we expect to solve the similar problem in $\mathbb{R}^3$. However, before we can accomplish this task we shall need a full classification of the QSICs in $\mathbb{R}^3$, which might be the first step in extending.

6. Acknowledgements

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