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An Optimal Lower Bound for Interval Routing in General Networks*

Savio S.H. Tse and Francis C.M. Lau†
Department of Computer Science and Information Systems
The University of Hong Kong
{sshtse,fcmlau}@csis.hku.hk

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Abstract

Interval routing is a space-efficient (compact) routing method for point-to-point communication networks. The method is based on proper labeling of edges of the graph with intervals. An optimal labeling would result in routing of messages through the shortest paths. Optimal labelings exist for regular as well as some of the common topologies, but not for arbitrary graphs. It has been shown that it is impossible to find optimal labelings for arbitrary graphs [4]. In this paper, we prove the lower bound of $2D - 3$ on the longest routing path for arbitrary graphs, where $D = O(\sqrt{n})$ is the graph's diameter and n is the number of nodes, as well as a lower bound of $2D - o(D)$ for $D = O(n)$. Our results are very close to the best known upper bound which is $2D$.
Keywords : interval routing, longest path, disjoint interval, cyclic order, chains.

1 Introduction

Interval routing was first proposed by Santoro and Khatib [6], and subsequently refined by van Leeuwen and Tan [11]. The idea is to label the nodes by integers (called node numbers) from a cyclicly ordered set, say, $\{0, 1, \ldots, n-1\}$, where n is the number of nodes; and the edges by intervals of the form $\langle p, q\rangle$, where p, q are node numbers. $\langle p, q\rangle$ is the set $\{p, p+1, \ldots, q\}$ if $p < q$, or $\{p, p+1, \ldots, n-1, 0, \ldots, q\}$ if $p > q$. $\langle p\rangle$ is the short form for $\langle p, p\rangle$, *i.e.*, the set $\{p\}$. During routing,

*A preliminary version of this paper is [9].

†Correspondence: Dr F.C.M. Lau, Department of Computer Science and Information Systems, The University of Hong Kong, Hong Kong / Email: fcmlau@csis.hku.hk / Fax: (+852) 2858 4141.

a message is routed along an edge whose interval label contains the destination node number, until the message reaches the destination. An example of interval routing is shown in Figure 1. The figure shows the routing path of a message that

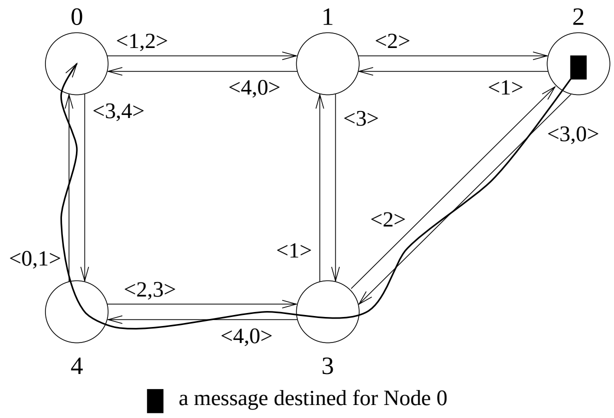

Figure 1: An example of interval routing

travels from Node 2 to Node 0. The message first takes the edge to Node 3 because 0 is contained in the interval $\langle 3, 0\rangle$, and then takes the edge to Node 4 because 0 is contained in $\langle 4, 0\rangle$, and so on. Clearly, with interval routing, at most $O(d)$ space is needed at a node, where d is the node's degree. In general, d is smaller than n, the size of the network, and we say that the routing information stored at a node as required by interval routing is "compact". See the survey by Tan and van Leeuwen [7] for an overview of the field of compact routing.

One of the main questions in interval routing research is that given G, how to label its nodes and edges so that all the routing paths are shortest paths, where G represents either a specific kind of graphs or arbitrary graphs (general networks). A successful labeling satisfying the condition constitutes an optimum interval routing scheme (IRS). For a number of specific graphs, optimum IRSs are known to exist [7]. What about arbitrary graphs? Ružička answered this in the negative way by constructing a graph that has no optimum IRS [4].

In practice, it might not always be necessary to insist on shortest-path routing, as long as the paths are not too far from the optimal. Santoro and Khatib have proposed an algorithm that can label any graph to yield paths whose lengths are at most two times the graph's diameter [6]. Instead of considering all the paths, we could look at just the longest path which is commonly used as a performance indicator in many analyses. In shortest-path routing, the longest path equals the diameter of the network. In other cases, it is useful to establish a lower bound in terms of the network's diameter on the longest path. This bound can then be used to determine the goodness of any routing scheme to be applied to the network. The aim of this paper is to present a lower bound on the longest path in 1-label interval routing.

Interestingly, only one upper bound exists, which is the $2D$ upper bound for 1-IRS according to Santoro and Khatib [6]. In this paper, the lower bounds are $2D-3$ and $2D - o(D)$, improving the result $\frac{7}{4}D - 1$ in [8]. Since there is no any better algorithm yielding an upper bound less than $2D$, and no any lower bound higher than $2D - 3$ there are still rooms for improvement on both sides.

2 Properties

The network in question is a connected graphs, $G = (V, E)$. where V is the set of nodes, and E the set of the edges. Every edge in E is bidirectional. There are n nodes in V. To implement interval routing, each node is labeled with a unique node number, from the set $\{0, \ldots, n-1\}$, and every edge in each direction by an interval called the edge's *interval label*. For $u, v \in V$ that are directly connected, $\mathcal{L}(u, v)$ denotes the interval label for the edge that goes from u to v.

An interval $\langle a, b\rangle$ is the set $a, a+1, \ldots, b \pmod n$. We refer to such a set an *interval set*. A set A is not an interval if and only if A is a proper subset of every interval set containing it. If an interval B contains an interval B', B' is called a *subinterval* of B.

We use the notation $u \prec v \prec w$, to denote the cyclic ordering of node numbers, for $u, v, w \in \{0, \ldots, n-1\}$. Naturally, $0 \prec 1 \prec \ldots \prec n-1 \prec 0$. As in [8], the expression $u \prec \{v, w\} \prec x$ means that v and w are contained in some interval and that they are ordered after u and before x, but the order of v and w is not shown.

Property 1 *(Completeness) The set of interval labels for edges directed from a node u is complete. That is, $\forall u \in V$, $V - \{u\} \subset \cup_{(u,v)\in E}\mathcal{L}(u, v)$.*

Property 2 *(No ambiguity) The interval labels for edges directed from a node u are disjoint. That is, for $u \neq v$, v is contained in exactly one of these intervals.*

Property 3 *(No bouncing) For each $(u, v) \in E$, there exists no node $w \neq u, v$, such that w is contained in both $\mathcal{L}(u, v)$ and $\mathcal{L}(v, u)$.*

Property 4 *(Lossless) Given a chain of nodes $w_1, w_2, \ldots, w_k$. $\forall i \in [1, k-1]$, if $u \neq w_j\, \forall j \in [i, k-1]$ and $u \in \mathcal{L}(w_i, w_{i+1})$, we have $u \in \mathcal{L}(w_i, w_{i+1}) \cap \ldots \cap \mathcal{L}(w_{k-1}, w_k)$.*

Property 5 *(Reachable) Given a chain of nodes $w_1, w_2, \ldots, w_k$. $\forall r, s, t \in [2, k-1]$, $r < s < t$, w_s is contained in $\mathcal{L}(w_r, w_{r+1}) \cup \mathcal{L}(w_{t+1}, w_t)$.*

Property 4 and 5 are referred to the interval labels of edges in a chain. For a chain a, b, c, d, e, Property 4 guarantees that $\forall x \in \mathcal{L}(a, b)$, x is in $\mathcal{L}(b, c)$ if $x \neq b$; and x is in $\mathcal{L}(c, d)$ if $x \neq b, c$, etc; otherwise, either the routing information about x is lost or Property 3 is violated. For the same chain, Property 5 guarantees that c, say, is included in $\mathcal{L}(a, b)$ or $\mathcal{L}(e, d)$; otherwise, c is not reachable. It should be noted that these 5 properties are necessary but not sufficient for a valid IRS for general graphs. A valid IRS is one that can route a message from any node to any other node. If Property 3 is changed to ”No cycle” instead of ”No bouncing”, these properties is sufficient for a valid IRS. We need not impose the Property of No Cycle because we will prove our bound by contradiction to our assumption that every routing path is shorter than $2D - K$, $K \geq 3$. By the structure of the graph used in our proof, it is impossible to have any cycle in a routing path with length not greater than $2D$. Hence, under our assumption on the longest path, the properties are sufficient for a valid IRS in the graph used below.

3 The Graph $G_{L,C,F}$

We define a graph $G_{L,C,F}$, as shown in Figure 2, based on which we prove our lower bound. Define $G_{L,C,F} = (V_{L,C,F}, E_{L,C,F})$ which is of diameter $D = 2C + 2$,

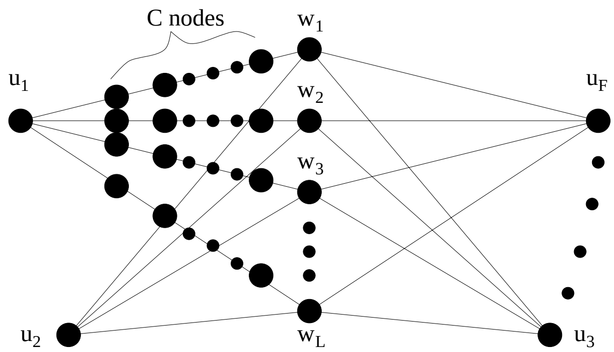

Figure 2: The skeleton of $G_{L,C,F}$.

and size $n = LCF + L + F$; $V_{L,C,F}$ and $E_{L,C,F}$ are as follows.

$$
\begin{aligned}
V_{L,C,F} = \;& \{v_{l,c,f} | 1 \leq l \leq L, 1 \leq c \leq C, 1 \leq f \leq F\} \\
\cup \;& \{u_f | 1 \leq f \leq F\} \\
\cup \;& \{w_l | 1 \leq l \leq L\} \\
E_{L,C,F} = \;& \{(v_{l,c,f}, v_{l,c+1,f}) | 1 \leq l \leq L, 1 \leq c \leq C-1, 1 \leq f \leq F\} \\
\cup \;& \{(u_f, v_{l,1,f}) | 1 \leq l \leq L, 1 \leq f \leq F\} \\
\cup \;& \{(w_l, v_{l,C,f}) | 1 \leq l \leq L, 1 \leq f \leq F\}
\end{aligned}
$$

There are F flaps, whose roots are the nodes $u_1, u_2, \ldots, u_F$, and within each flap, C columns and L layers. We will prove the lower bound by contradiction. The values of L, C and F will be defined later. We use the subscripts l, c, f to denote the layer, the column, and the flap, respectively.

Definition 1 *An lf-chain (or simply chain) is the set $\{v_{l,1,f}, v_{l,2,f}, \ldots, v_{l,C,f}\}$.*

Example of Definition 1: a 32-chain is the set $\{v_{3,1,2}, v_{3,2,2}, \ldots, v_{3,C,2}\}$, and a $(12)(91)$-chain is the set $\{v_{12,1,91}, v_{12,2,91}, \ldots, v_{12,C,91}\}$. The nodes of a chain may fall into one or more disjoint intervals. Two chains are disjoint if their nodes fall into two disjoint intervals, respectively. Similarly, two layers (or flaps) are disjoint if the nodes of their chains fall into two disjoint intervals, respectively.

For the edge labels $\mathcal{L}(u_f, v_{l,1,f})$ and $\mathcal{L}(w_l, v_{l,C,f})$ at the end points of an lf-chain, we have two cases. The first case is that the union of the edge labels is an interval; and the second case is just the opposite. We need to divide all the chains into two cases because our proof is mainly based on the first case, and on the other hand, we can prove the number of chains of the second case is bounded (Lemma 4.2). The chains of the first case are called *normal* because the number of them is unbounded; and the other chains are, therefore, called *abnormal*.

Definition 2 *An lf-chain is normal if $\mathcal{L}(u_f, v_{l,1,f}) \cup \mathcal{L}(w_l, v_{l,C,f})$ is an interval. A chain which is not normal is said to be abnormal.*

Lemma 3.1 *The union of two non-disjoint intervals is an interval.* $\diamondsuit$

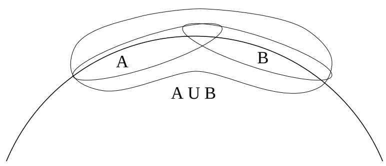

Figure 3: Idea of Lemma 3.1.

Lemma 3.2 *If an lf-chain is abnormal, $\mathcal{L}(u_f, v_{l,1,f}) \cap \mathcal{L}(w_l, v_{l,C,f}) = \emptyset$.*

Proof: Directly by Lemma 3.1. $\diamondsuit$

Lemma 3.3 *If an lf-chain is abnormal, $\exists a, b \notin \mathcal{L}(u_f, v_{l,1,f}) \cup \mathcal{L}(w_l, v_{l,C,f})$ such that $a \prec \mathcal{L}(u_f, v_{l,1,f}) \prec b \prec \mathcal{L}(w_l, v_{l,C,f}) \prec a$.*

Proof: Directly by Definition 2. $\diamondsuit$

Hierachery of lemmas and theorems.

The main results are Theorem 7.1 and Theorem 7.2, which are direct consequence from Theorem 6.1. We use counter proof technique and assume every path is no longer than $2D - K$. To prove Theorem 6.1, we need the lemmas. The main Lemma—Lemma 5.1 proves that there is at least a path no shorter than $\frac{3}{2}D - 1$ if we allow partial disjointness of two layers (labeled as "A" in Figure 4). However,

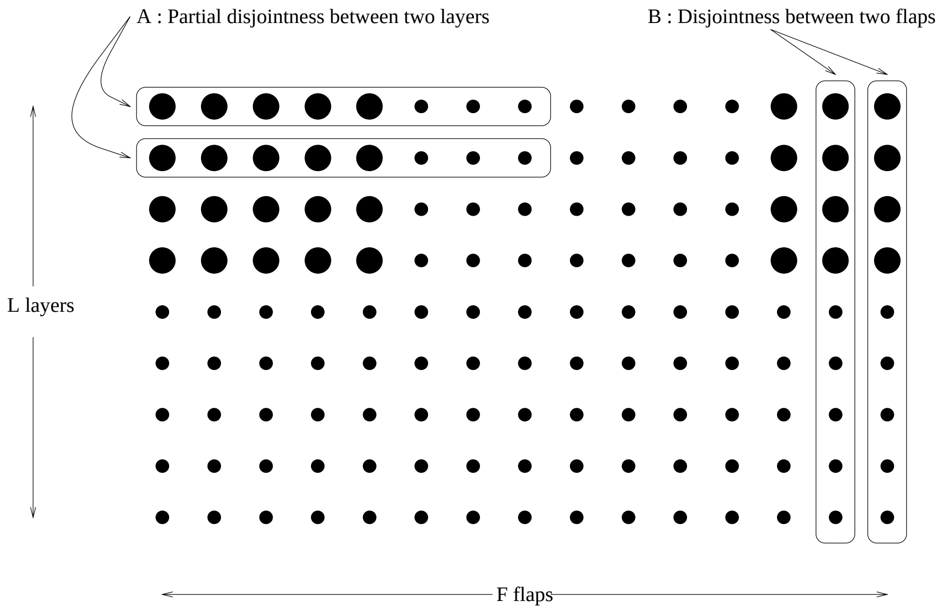

Figure 4: A matrix of $v_{l,1,f}, \forall l \in [1, L], f \in [1, F]$.

such a path of length $\frac{3}{2}D - 1$ will result a path no shorter than $2D - 1$ (Theorem 6.1) which violates our assumption on the longest path length. Hence, such partial disjointness between layers (Lemma 5.1) cannot exist. However, the absence of this partial disjointness between layers will lead to the presence of disjointness between two flaps (labeled as "B" in Figure 4). Lastly, the presence of disjointness between two flaps will lead to the violation of the assumption on longest path length (Theorem 6.1).

Most of the dependencies of theorems and lemmas are shown in Figure 5 which may increase the readability of this proof.

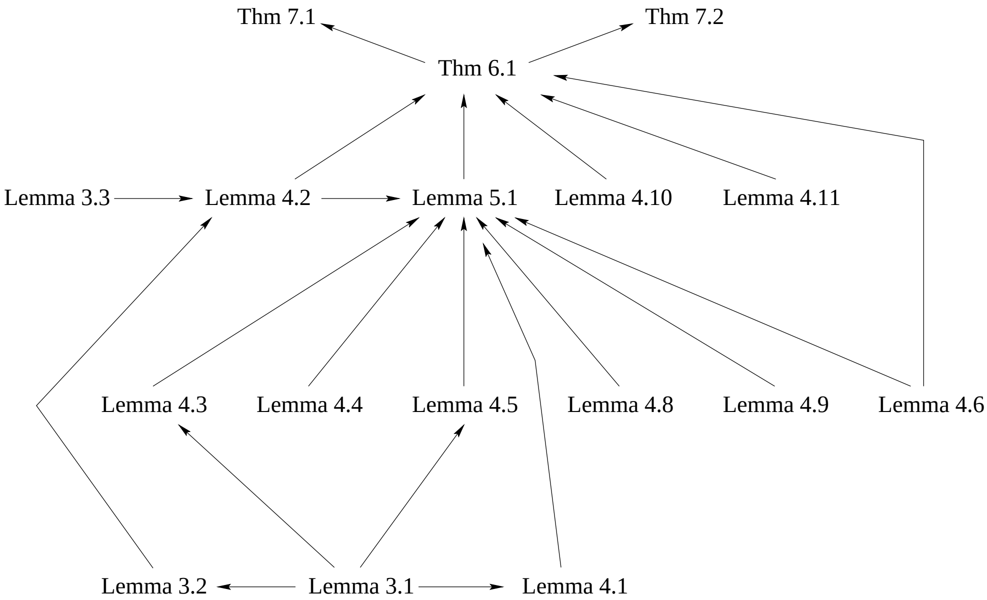

Figure 5: Hierachery of theorems and lemmas.

4 Lemmas on Chains

The Lemmas from 4.1 to 4.11, hold under the assumption that there exists a labeling scheme such that every path is shorter than $2D - K$, $K \geq 3$. The results of these lemmas are that in any layer, there are at least $F - \lfloor \frac{D}{K} \rfloor - 2$ normal chains. The main lemma—Lemma 5.1 will hold for sufficient number of normal chains which can be obtained from a good choice of the value of F. The lemmas, from Lemma 4.1 onwards, will be used in the proof of main Lemma 5.1, and in the proof of Theorem 6.1.

Lemma 4.1 $\forall\, l \in [1, L], f \in [1, F]$, *an lf-chain can be partitioned into at most* 4 *interval sets,* $W_{l,f}$, $X_{l,f}$, $Y_{l,f}$, $Z_{l,f}$, *such that* $a_1 \prec W_{l,f} \prec a_2 \prec X_{l,f} \prec a_3 \prec Y_{l,f} \prec a_4 \prec Z_{l,f} \prec a_1$ *where* $a_i \in V_{L,C,F} - \{v_{l,1,f}, v_{l,2,f}, \ldots, v_{l,C,f}\}$, $i \in [1, 4]$.

Proof: Assume there are at least 5 interval sets $W_{l,f}$,$X_{l,f}$, $Y_{l,f}$, $Z_{l,f}$, $T_{l,f}$ which partition the lf-chain such that $a_1 \prec W_{l,f} \prec a_2 \prec X_{l,f} \prec a_3 \prec Y_{l,f} \prec a_4 \prec Z_{l,f} \prec a_5 \prec T_{l,f} \prec a_1$ where $a_i \in V_{L,C,F} - \{v_{l,1,f}, v_{l,2,f}, \ldots, v_{l,C,f}\}$, $i \in [1, 5]$.

$W_{l,f}, X_{l,f}, Y_{l,f}, Z_{l,f}, T_{l,f} \subset \mathcal{L}(u_f, v_{l,1,f}) \cup (\cup_{c=1}^{c=C-1} \mathcal{L}(v_{l,c,f}, v_{l,c+1,f}))$. Since $\mathcal{L}(u_f, v_{l,1,f})$ and $\mathcal{L}(v_{l,1,f}, v_{l,2,f})$ are not disjoint (because $v_{l,2,f} \in \mathcal{L}(u_f, v_{l,1,f}) \cap \mathcal{L}(v_{l,1,f}, v_{l,2,f})$ under the assumption on the longest path), the union of $\mathcal{L}(u_f, v_{l,1,f})$ and $\mathcal{L}(v_{l,1,f}, v_{l,2,f})$ is an interval, by Lemma 3.1. Moreover, $\forall c \in [1, C-2]$, $\mathcal{L}(v_{l,c,f}, v_{l,c+1,f})$ and $\mathcal{L}(v_{l,c+1,f}, v_{l,c+2,f})$ are not disjoint, $\mathcal{L}(u_f, v_{l,1,f}) \cup (\cup_{c=1}^{c=C-1} \mathcal{L}(v_{l,c,f}, v_{l,c+1,f}))$ is an interval. Then 4 of $a_1, \ldots, a_5$ are contained in $\mathcal{L}(u_f, v_{l,1,f}) \cup (\cup_{c=1}^{c=C-1} \mathcal{L}(v_{l,c,f}, v_{l,c+1,f}))$, and by Property 4, these 4 elements are contained in $\mathcal{L}(v_{l,C,f}, w_l)$. Then, by Property 3, $\mathcal{L}(w_l, v_{l,C,f})$ will contain at most 2 of $W_{l,f}, X_{l,f}, Y_{l,f}, Z_{l,f}, T_{l,f}$.

Similarly, $\mathcal{L}(u_f, v_{l,1,f})$ will also contain at most 2 of $W_{l,f}, X_{l,f}, Y_{l,f}, Z_{l,f}, T_{l,f}$. Then, at least one of $W_{l,f}, X_{l,f}, Y_{l,f}, Z_{l,f}, T_{l,f}$ is not a subset of $\mathcal{L}(u_f, v_{l,1,f}) \cup \mathcal{L}(w_l, v_{l,C,f})$. Property 5 is violated. $\diamondsuit$

Lemma 4.2 *In a layer, there are at most* $\lfloor \frac{D}{K} \rfloor + 2$ *abnormal chains.*

Proof: Assume in the lth layer, there are p chains in f_1th, f_2th,…,f_pth flaps, such that $\forall i \in [1, p]$, $\mathcal{L}(u_{f_i}, v_{l,1,f_i}) \cup \mathcal{L}(w_l, v_{l,C,f_i})$ is not an interval. For all $i \in [1, p]$, let $x_i = \max\{c | v_{l,c,f_i} \in \mathcal{L}(u_{f_i}, v_{l,1,f_i})\}$ and $y_i = \min\{c | v_{l,c,f_i} \in \mathcal{L}(w_l, v_{l,C,f_i})\}$.

For all $i \in [1, p]$, if $y_i > x_i + 1$, we have $v_{l,x_i+1,f_i} \notin \mathcal{L}(u_{f_i}, v_{l,1,f_i}) \cup \mathcal{L}(w_l, v_{l,C,f_i})$; contradicting Property 5. Hence, $y_i \leq x_i + 1$.

$\forall r, s \in [1, C], 0 < r - s \leq 2K - 1$, we want to argue that there are at most 2 elements, say, $i, j \in [1, p]$, such that $y_i, y_j \in [s, r]$. Assume the contrary, there are 3 elements, say, $i, j, k \in [1, p]$, such that $y_i, y_j, y_k \in [s, r]$. Consider x_i, y_i. $v_{l,x_i,f_i} \in \mathcal{L}(u_{f_i}, v_{l,1,f_i})$ implies $v_{l,x_i,f_i} \in \mathcal{L}(v_{l,c-1,f_i}, v_{l,c,f_i}), \forall c \leq x_i$. By the assumption on the path lengths, $v_{l,y_i,f_i} \in \mathcal{L}(v_{l,c-1,f_i}, v_{l,c,f_i}), \forall c \in [y_i - K + 1, y_i]$. Since $y_i \leq x_i + 1$, there exists a $c \in [y_i - K + 1, y_i - 1]$ such that $v_{l,x_i,f_i}, v_{l,y_i,f_i} \in \mathcal{L}(v_{l,c-1,f_i}, v_{l,c,f_i})$. Similarly, there exists a $c' \in [x_i + 1, x_i + K - 1]$ such that $v_{l,x_i,f_i}, v_{l,y_i,f_i} \in \mathcal{L}(v_{l,c'+1,f_i}, v_{l,c',f_i})$. Obviously, $c < c'$. By Property 3 and 4, $\mathcal{L}(v_{l,c-1,f_i}, v_{l,c,f_i}) \cap \mathcal{L}(v_{l,c'+1,f_i}, v_{l,c',f_i})$ will contain elements only in $\{v_{l,c,f}, v_{l,c+1,f}, \ldots, v_{l,c',f}\}$. Hence, we have the cyclic structure in Figure 6 due to the presence of a, b (also in the figure) by Lemma 3.3, where $a, b \notin \mathcal{L}(u_{f_i}, v_{l,1,f_i}) \cup \mathcal{L}(w_l, v_{l,C,f_i})$.

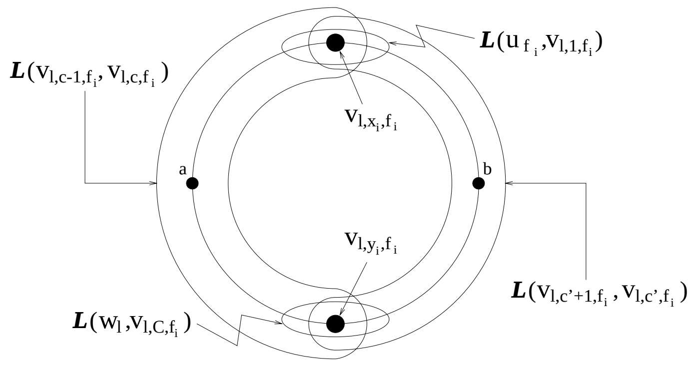

Figure 6: Cyclic structure of $\mathcal{L}(v_{l,c-1,f_i}, v_{l,c,f_i})$ and $\mathcal{L}(v_{l,c'+1,f_i}, v_{l,c',f_i})$.

Consider x_j, y_j. Like x_i and y_i, $y_j \leq x_j + 1$. We now want to find the positions of v_{l,x_j,f_j} and v_{l,y_j,f_j} in the cyclic structure as in Figure 6. By Lemma 3.2, $\mathcal{L}(u_{f_j}, v_{l,1,f_j}) \cap \mathcal{L}(w_l, v_{l,C,f_j}) = \emptyset$. Therefore, $v_{l,x_j,f_j} \notin \mathcal{L}(w_l, v_{l,C,f_j})$ and $v_{l,y_j,f_j} \notin \mathcal{L}(u_{f_j}, v_{l,1,f_j})$.

If $v_{l,x_j,f_j} \in \mathcal{L}(v_{l,c-1,f_i}, v_{l,c,f_i})$, the routing path from $v_{l,c-1,f_i}$ to v_{l,x_j,f_j} will passes through $v_{l,c-1,f_i}, w_l, u_{f'}, w_{l'}, u_{f_j}, v_{l,x_j,f_j}$. Choosing $c = y_i - K + 1$, the routing path length is $(\frac{D}{2} - (c-1)) + \frac{D}{2} + \frac{D}{2} + \frac{D}{2} + x_j = 2D - (c-1) + x_j \geq 2D - (y_i - K) + (y_j - 1) = 2D - (y_i - y_j) + K - 1 \geq 2D - (2K-1) + K - 1 = 2D - K$. Contradiction to the assumption on the path length. Hence, $v_{l,x_j,f_j} \in \mathcal{L}(v_{l,c'+1,f_i}, v_{l,c',f_i})$.

If $v_{l,y_j,f_j} \in \mathcal{L}(v_{l,c'+1,f_i}, v_{l,c',f_i})$, the routing path from $v_{l,c'+1,f_i}$ to v_{l,y_j,f_j} will passes through $v_{l,c'+1,f_i}, u_{f_i}, w_{l'}, u_{f'}, w_l, v_{l,y_j,f_j}$. Choosing $c' = x_i + K - 1$, the routing path length is $(c'+1) + \frac{D}{2} + \frac{D}{2} + \frac{D}{2} + (\frac{D}{2} - y_j) = 2D + (c'+1) - y_j = 2D + (x_i + K) - y_j \geq 2D + (y_i + K - 1) - y_j = 2D - (y_j - y_i) + K - 1 \geq 2D - (2K-1) + K - 1 = 2D - K$. Contradiction to the assumption on the path length. Hence, $v_{l,y_j,f_j} \in \mathcal{L}(v_{l,c-1,f_i}, v_{l,c,f_i})$.

Similarly, $v_{l,x_k,f_k} \in \mathcal{L}(v_{l,c'+1,f_i}, v_{l,c',f_i})$ and $v_{l,y_k,f_k} \in \mathcal{L}(v_{l,c-1,f_i}, v_{l,c,f_i})$, as in Figure 7(a).

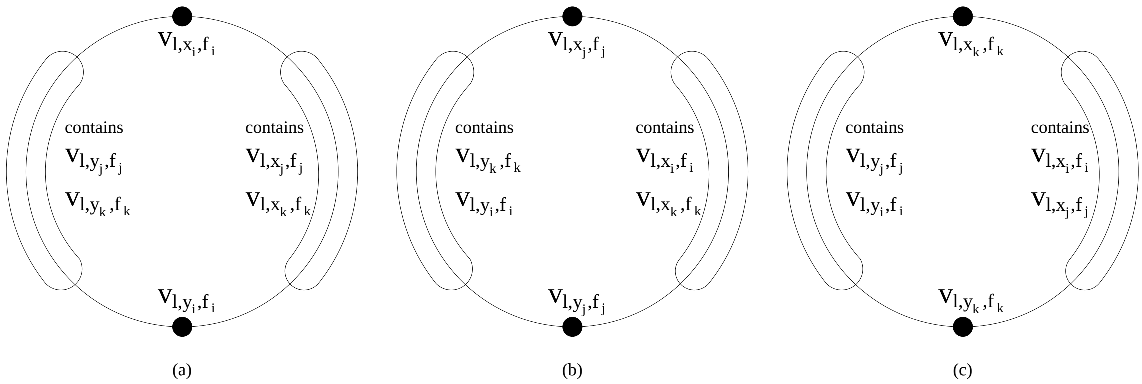

Figure 7: Cyclic structure of $v_{l,x_i,f_i}, v_{l,y_i,f_i}, v_{l,x_j,f_j}, v_{l,y_j,f_j}, v_{l,x_k,f_k}$, and v_{l,y_k,f_k}.

Since, f_ith, f_jth, f_kth flaps are symmetric, we have another two cases of cyclic structures as in Figure 7(b)(c). Obviously, the three cases in Figure 7(a)(b)(c) will contradict to one another.

Hence, $\forall r, s \in [1, C], 0 < r - s \leq 2K - 1$, there are at most 2 elements, say, $i, j \in [1, p]$, such that $y_i, y_j \in [s, r]$. Therefore, $p \leq 2\lceil \frac{D}{2K} \rceil < \lfloor \frac{D}{K} \rfloor + 2$. $\diamondsuit$

Lemma 4.3 *$\forall l \in [1, L], f \in [1, F]$, if an lf-chain is partitioned into 4 interval sets, $W_{l,f}$, $X_{l,f}$, $Y_{l,f}$ and $Z_{l,f}$, such that $a_1 \prec W_{l,f} \prec a_2 \prec X_{l,f} \prec a_3 \prec Y_{l,f} \prec a_4 \prec Z_{l,f} \prec a_1$ where $a_i \in V_{L,C,F} - \{v_{l,1,f}, v_{l,2,f}, \ldots, v_{l,C,f}\}$, $i \in [1, 4]$, the lf-chain is abnormal.*

Proof: For $l \in [1, L], f \in [1, F]$, assume there are 4 interval sets $W_{l,f}$,$X_{l,f}$, $Y_{l,f}$, $Z_{l,f}$ which partition the lf-chain such that $a_1 \prec W_{l,f} \prec a_2 \prec X_{l,f} \prec a_3 \prec Y_{l,f} \prec a_4 \prec Z_{l,f} \prec a_1$ where $a_i \in V_{L,C,F} - \{v_{l,1,f}, v_{l,2,f}, \ldots, v_{l,C,f}\}$, $i \in [1, 4]$.

By using similar argument as in the proof of Lemma 4.1, at least 3 of $a_1, \ldots, a_4$ are contained in $\mathcal{L}(u_f, v_{l,1,f}) \cup (\cup_{c=1}^{c=C-1} \mathcal{L}(v_{l,c,f}, v_{l,c+1,f}))$, and by Property 4, these

3 elements are contained in $\mathcal{L}(v_{l,C,f}, w_l)$. By Property 3, at most one element of $a_1, \ldots, a_4$ is contained in $\mathcal{L}(w_l, v_{l,C,f})$. Hence, $\mathcal{L}(w_l, v_{l,C,f})$ can only contain at most 2 of $W_{l,f}, X_{l,f}, Y_{l,f}, Z_{l,f}$, say, $W_{l,f}$ and $X_{l,f}$ (or $\{X_{l,f}, Y_{l,f}\}$ or $\{Y_{l,f}, Z_{l,f}\}$ or $\{Z_{l,f}, W_{l,f}\}$). Similarly, $\mathcal{L}(u_f, v_{l,1,f})$ can only contain at most 2 of $W_{l,f}, X_{l,f}, Y_{l,f}, Z_{l,f}$, too. Hence, $\mathcal{L}(u_f, v_{l,1,f})$ will only contain $Y_{l,f}, Z_{l,f}$, which are not contained by $\mathcal{L}(w_l, v_{l,C,f})$, by Property 5.

Recall that both $\mathcal{L}(u_f, v_{l,1,f})$ and $\mathcal{L}(w_l, v_{l,C,f})$ contain at most one element from $a_1, \ldots, a_4$, and according to the given cyclic structure, $\mathcal{L}(u_f, v_{l,1,f})$ contains a_4 and $\mathcal{L}(w_l, v_{l,C,f})$ contains a_2. The cyclic structure is

$$a_1 \prec \mathcal{L}(w_l, v_{l,C,f}) \prec a_3 \prec \mathcal{L}(u_f, v_{l,1,f}) \prec a_1.$$

Hence, $\mathcal{L}(w_l, v_{l,C,f}) \cup \mathcal{L}(u_f, v_{l,1,f})$ is not an interval, because it can contain neither a_1 nor a_3. Then, the lf-chain is abnormal. $\diamondsuit$

Hereafter, if a chain is said to have exactly k interval sets, there exist k intervals which only contain all elements in the chain and any $k-1$ intervals contain all elements in the chain will contain at least one element in other chain. Lemma 4.1 shows that $k \leq 4$ for all chains. Lemma 4.2 shows that the abnormal chains are the minority. Lemma 4.3 shows that $k \leq 3$ for normal chains. So, we have 3 cases. Lemma 4.4 is for the chains having exactly 1 interval set; Lemma 4.5 and 4.6 are for the chains having exactly 2 interval sets; and Lemma 4.7, 4.8, 4.9 and 4.11 are for the chains having 3 interval sets.

Lemma 4.4 *Given an $l \in [1, L]$ and an interval A containing $\{v_{l,C,f} | f \in [1, F']\}$, where $3 \leq F' \leq F$, and $b \prec v_{l,C,1} \prec v_{l,C,2} \prec \cdots \prec v_{l,C,F'} \prec b$, where $b \notin A$. If an lf-chain, $f \in [2, F'-1]$, has exactly one interval set $X_{l,f}$, and if all routing paths from w_l through $v_{l,C,f'}$ is shorter than $\frac{3}{2}D - 1$, $\forall v_{l,C,f'} \in A$ where $f' \in [1, F]$, we have either*

$$\begin{array}{rl} & \cdots \prec v_{l,C,f-1} \prec H \prec v_{l,1,f} \prec v_{l,C,f} \prec v_{l,C,f+1} \prec \cdots \\ \textit{or} & \cdots \prec v_{l,C,f-1} \prec v_{l,C,f} \prec v_{l,1,f} \prec H \prec v_{l,C,f+1} \prec \cdots \end{array}$$

where the set $H = \mathcal{L}(w_l, v_{l,C,f}) - X_{l,f}$.

Proof: By the assumption on the length path, all paths will be shorter than $2D - K$, $K \geq 3$. Therefore, $\forall f'' \in [1, F]$, $v_{l,C,f''} \in \mathcal{L}(w_l, v_{l,C,f''})$. Then,

- $v_{l,1,f} \in \mathcal{L}(w_l, v_{l,C,f})$.
  Assume $v_{l,1,f} \notin \mathcal{L}(w_l, v_{l,C,f})$, $\exists f''' \neq f$ such that $v_{l,1,f} \in \mathcal{L}(w_l, v_{l,C,f'''})$, by Property 1 on w_l. If $v_{l,C,f'} \notin A$, $\mathcal{L}(w_l, v_{l,C,f'})$ will contain at least $v_{l,C,1}$ or $v_{l,C,F'}$, because $v_{l,1,f} \in X_{l,f}$, *ie.* $v_{l,1,f}$ is a non-marginal element in A, and

hence, a contradiction to Property 2 on w_l. Therefore, $v_{l,C,f'} \in A$ and the routing path length from w_l to $v_{l,1,f}$ through $v_{l,C,f'} \in A$ is longer than $\frac{3}{2}D - 1$. Contradiction.

- $v_{l,C,f-1} \prec v_{l,1,f} \prec v_{l,C,f} \prec v_{l,C,f+1} \Rightarrow v_{l,C,f-1} \prec H \prec v_{l,1,f} \prec v_{l,C,f} \prec v_{l,C,f+1}$.

  Assume $\exists h \in H$ such that $v_{l,C,f-1} \prec v_{l,1,f} \prec v_{l,C,f} \prec h \prec v_{l,C,f+1}$. By Property 4, $\mathcal{L}(v_{l,C-1,f}, v_{l,C-2,f})$ will contain h and $v_{l,1,f}$. By the longest path assumption (*ie.* $2D - K$), $\mathcal{L}(v_{l,C-1,f}, v_{l,C,f})$ will contain $v_{l,C,f-1}$, $v_{l,C,f}$ and $v_{l,C,f+1}$. So the underlined nodes of

  $$\underline{v_{l,C,f-1}} \prec v_{l,1,f} \prec \underline{v_{l,C,f}} \prec h \prec \underline{v_{l,C,f+1}}$$

  are belonged to $\mathcal{L}(v_{l,C-1,f}, v_{l,C,f})$, and $h, v_{l,1,f}$ are belonged to $\mathcal{L}(v_{l,C-1,f}, v_{l,C-2,f})$. Contradiction to Property 2 on $v_{l,C-1,f}$.

The other case— $v_{l,C,f-1} \prec v_{l,C,f} \prec v_{l,1,f} \prec v_{l,C,f+1} \Rightarrow v_{l,C,f-1} \prec v_{l,C,f} \prec v_{l,1,f} \prec h \prec v_{l,C,f+1}$ is just similar to the above case. Hence, result follows. $\diamondsuit$

Lemma 4.5 *$\forall l \in [1, L], f \in [1, F]$, assume an lf-chain has exactly two interval sets $X_{l,f}$ and $Y_{l,f}$. Let $A = \mathcal{L}(w_l, v_{l,C,f}) - X_{l,f} - Y_{l,f}$ and $B = \mathcal{L}(u_f, v_{l,1,f}) - X_{l,f} - Y_{l,f}$. We have either*

$$\begin{array}{rl} & X_{l,f} \prec B \prec Y_{l,f} \prec A \prec X_{l,f} \\ \textit{or} & X_{l,f} \prec A \prec Y_{l,f} \prec B \prec X_{l,f} \end{array}$$

Proof: Since $X_{l,f} \cup Y_{l,f}$ is not an interval, $X_{l,f}$ and $Y_{l,f}$ divide the set $V_{L,C,F} - X_{l,f} - Y_{l,f}$ into 2 intervals, say, P and Q. Without loss of generality, we have the cyclic structure $P \prec X_{l,f} \prec Q \prec Y_{l,f} \prec P$.

Using similar technique as the proof of Lemma 4.3, we can prove that at least one of P and Q is contained in $\mathcal{L}(u_f, v_{l,1,f}) \cup (\cup_{c=1}^{c=C-1} \mathcal{L}(v_{l,c,f}, v_{l,c+1,f}))$ which is an interval, by Lemma 3.1. Hence, at least one of P and Q is contained in $\mathcal{L}(v_{l,C,f}, w_l)$, by Property 4. By Property 3, at most one of P and Q will have intersection with $\mathcal{L}(w_l, v_{l,C,f})$. In other words, if $\mathcal{L}(w_l, v_{l,C,f})$ contains elements not in the lf-chain, these elements will be in either P or Q, say P.

Assume $\mathcal{L}(u_f, v_{l,1,f})$ contains elements which is also in P. Let $p_u \in \mathcal{L}(u_f, v_{l,1,f}) \cap P$ and $p_w \in \mathcal{L}(w_l, v_{l,C,f}) \cap P$. So, by Property 4, we have $p_u \in \mathcal{L}(u_f, v_{l,1,f}) \cap (\cap_{c=1}^{c=C-1} \mathcal{L}(v_{l,c,f}, v_{l,c+1,f}))$ and $p_w \in \mathcal{L}(w_l, v_{l,C,f}) \cap (\cap_{c=2}^{c=C} \mathcal{L}(v_{l,c,f}, v_{l,c-1,f}))$. And, by Property 3, we have $p_w \notin \mathcal{L}(u_f, v_{l,1,f}) \cup (\cup_{c=1}^{c=C-1} \mathcal{L}(v_{l,c,f}, v_{l,c+1,f}))$ and $p_u \notin \mathcal{L}(w_f, v_{l,C,f}) \cup (\cup_{c=2}^{c=C} \mathcal{L}(v_{l,c,f}, v_{l,c-1,f}))$.

Since $X_{l,f} \cup Y_{l,f} = \{v_{l,1,f}, v_{l,2,f}, \ldots, v_{l,C,f}\}$, $\exists x \in X_{l,f}, y \in Y_{l,f}$ such that (x, y) is an edge. Without loss of generality, let $x = v_{l,c,f}$ and $y = v_{l,c+1,f}$. Since $x, y, p_u \in$

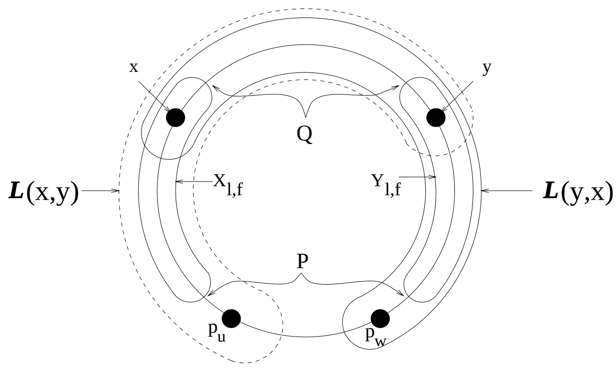

Figure 8: $Q \subset \mathcal{L}(x, y) \cap \mathcal{L}(y, x)$.

$\mathcal{L}(x, y)$ and $p_w \notin \mathcal{L}(x, y)$, $Q \subset \mathcal{L}(x, y)$, as in Figure 8. On the other hand, $x, y, p_w \in \mathcal{L}(y, x)$ and $p_u \notin \mathcal{L}(y, x)$, $Q \subset \mathcal{L}(y, x)$. Contradiction to Property 3. $\diamondsuit$

Lemma 4.6 *Given an $l \in [1, L]$ and an interval A containing $\{v_{l,C,f} | f \in [1, F']\}$, where $3 \leq F' \leq F$, and $b \prec v_{l,C,1} \prec \{v_{l,C,2}, v_{l,C,3}, \ldots, v_{l,C,F'-1}\} \prec v_{l,C,F'} \prec b$, where $b \notin A$. Assume the lf-chains are normal, $\forall f \in [1, F']$. Let $3 \leq T \leq F' - 2$. If there are T lf-chains, $f \in [2, F' - 1]$, having exactly two interval sets, we have at least $T - 2$ of these chains which are contained in A. Further, if the routing paths from w_l through $v_{l,C,f'}$ is shorter than $\frac{3}{2}D - 1$, $\forall v_{l,C,f'} \in A$ where $f' \in [1, F]$, then each of the above $T - 2$ lf-chains will be belonged to $\mathcal{L}(w_l, v_{l,C,f})$, respectively.*

Proof: Assume there are T lf-chains, $f \in [2, F' - 1]$, and each has exactly two interval sets. Without loss of generality, let these T chains be $l2$-chain, $l3$-chain,...,$l(T + 1)$-chain and let $Y_{l,f}$ be the interval set containing $v_{l,C,f}$, $\forall f \in [2, T + 1]$. Assume $Y_{l,2} \prec Y_{l,3} \prec \cdots \prec Y_{l,T+1} \prec Y_{l,2}$, then we have $b \prec v_{l,C,1} \prec Y_{l,2} \prec Y_{l,3} \prec \cdots \prec Y_{l,T+1} \prec v_{l,C,F'} \prec b$.

Let $X_{l,f}$ be another interval set of the lf-chain, $\forall f \in [2, T + 1]$. We now going to prove that there are at most 2 chains, say, lf_1-chain and lf_2-chain, $f_1, f_2 \in [2, T + 1]$, such that for $i = 1, 2$, $X_{l,f_i} \prec v_{l,C,1} \prec Y_{l,2} \prec Y_{l,3} \prec \cdots \prec Y_{l,T+1} \prec v_{l,C,F'} \prec X_{l,f_i}$. We will prove it by contradiction. Assume that there are at least 3 chains, say, lf_1-chain, lf_2-chain and lf_3-chain, $f_1, f_2, f_3 \in [2, T + 1]$, such that $X_{l,f_1} \prec X_{l,f_2} \prec X_{l,f_3} \prec v_{l,C,1} \prec Y_{l,2} \prec Y_{l,3} \prec \cdots \prec Y_{l,T+1} \prec v_{l,C,F'} \prec X_{l,f_1}$ as Figure 9.

Obviously, for $i = 1, 2, 3$, $\mathcal{L}(w_l, v_{l,C,f_i})$ will not contain any elements in X_{l,f_i}; otherwise, $\mathcal{L}(w_l, v_{l,C,f_i})$ will contain either $v_{l,C,1}$ or $v_{l,C,F'}$, contradicting to Property 2 on w_l. Hence, for $i = 1, 2, 3$, $\mathcal{L}(u_{f_i}, v_{l,1,f_i})$ will contain X_{l,f_i}, by Property 5. Since lf_2-chain (Figure 9) is normal, $\mathcal{L}(u_{f_2}, v_{l,1,f_2}) \cup \mathcal{L}(w_l, v_{l,C,f_2})$ is an interval and therefore, $\mathcal{L}(u_{f_2}, v_{l,1,f_2})$ will contain either X_{l,f_3} or X_{l,f_1}, say X_{l,f_1} as Figure 9. So,

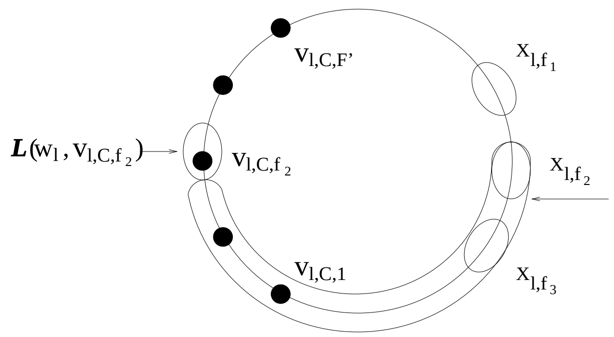

Figure 9: $\mathcal{L}(u_{f_2}, v_{l,1,f_2}) \cup \mathcal{L}(w_l, v_{l,C,f_2})$ is an interval.

the routing path from u_{f_2} to X_{l,f_1} will be $u_{f_2}, w_l, u_{f'}, w_{l'}, u_{f_1}, X_{l,f_1}$, where $f' \neq 2, f_1$ and $l' \neq l$. The length of this routing path is longer than $2D$ and contradicts to the assumption on the longest path.

Hence, there are at most 2 lf-chains, say, lf_1-chain and lf_2-chain, $f_1, f_2 \in [2, T+1]$, such that for $i = 1, 2$, $X_{l,f_i} \prec v_{l,C,1} \prec Y_{l,2} \prec Y_{l,3} \prec \cdots \prec Y_{l,T+1} \prec v_{l,C,F'} \prec X_{l,f_i}$. In other words, in the lth layer, we have at least $T-2$ lf-chains in $f_3, f_4, \ldots, f_{T+1}$ flaps such that $b \prec v_{l,C,1} \prec X_{l,f_j} \prec v_{l,C,F'} \prec b$, for $j \in [3, T+1]$.

We are now going to prove the disjoint property of these $T-2$ chains. Using the same argument in the first point of the proof of Lemma 4.4, $\mathcal{L}(w_l, v_{l,C,f})$ must contain every element in $Y_{l,f}$, $\forall f \in [2, T+1]$; otherwise, the assumption of path length from w_l ($\frac{3}{2}D - 1$) will be violated.

For $f \in [2, T+1]$, if the cyclic order is $b \prec v_{l,C,1} \prec X_{l,f} \prec v_{l,C,F'} \prec b$, $\mathcal{L}(w_l, v_{l,C,f})$ must contain $X_{l,f}$ (by the same argument in the first point of the proof of Lemma 4.4); otherwise, the assumption of path length from w_l ($\frac{3}{2}D - 1$) will be violated. Hence, $\mathcal{L}(w_l, v_{l,C,f})$ contains $X_{l,f} \cup Y_{l,f}$. $\diamondsuit$

Lemma 4.7 $\forall l \in [1, L], f \in [1, F]$, *if an lf-chain has exactly 3 interval sets $X_{l,f}$, $Y_{l,f}$, $Z_{l,f}$, we have the following :* (1) *among $X_{l,f}$, $Y_{l,f}$, $Z_{l,f}$, one (say $X_{l,f}$) will be a subset of $\mathcal{L}(u_f, v_{l,1,f}) - \mathcal{L}(w_l, v_{l,C,f})$,* (2) *another one (say $Z_{l,f}$) will be a subset of $\mathcal{L}(w_l, v_{l,C,f}) - \mathcal{L}(u_f, v_{l,1,f})$, and* (3) $\mathcal{L}(u_f, v_{l,1,f}) \cap \mathcal{L}(w_l, v_{l,C,f}) \subset Y_{l,f}$.

Proof: Using similar technique as the proof of Lemma 4.3, we can prove that both $\mathcal{L}(u_f, v_{l,1,f})$ and $\mathcal{L}(w_l, v_{l,C,f})$ can contain elements from at most 2 of $X_{l,f}$, $Y_{l,f}$ and $Z_{l,f}$. (The detail is left to the reader.) By Property 5, at least one of $X_{l,f}$, $Y_{l,f}$ and $Z_{l,f}$, say $X_{l,f}$, is a subset of $\mathcal{L}(u_f, v_{l,1,f}) - \mathcal{L}(w_l, v_{l,C,f})$. Similarly, at least one of $X_{l,f}$, $Y_{l,f}$ and $Z_{l,f}$, say $Z_{l,f}$, is a subset of $\mathcal{L}(w_l, v_{l,C,f}) - \mathcal{L}(u_f, v_{l,1,f})$. Hence, $\mathcal{L}(w_l, v_{l,C,f}) \cap \mathcal{L}(u_f, v_{l,1,f})$ will not contain element in $X_{l,f}$ and $Z_{l,f}$. If $\mathcal{L}(w_l, v_{l,C,f}) \cap \mathcal{L}(u_f, v_{l,1,f}) \neq \emptyset$. $\mathcal{L}(w_l, v_{l,C,f}) \cap \mathcal{L}(u_f, v_{l,1,f})$ will only contain elements in $\{v_{l,c,f}|c \in [1, C]\}$; otherwise, by Property 4, it is a contradiction to Property 3. Hence, it will only contain elements in $Y_{l,f}$. Therefore, $\mathcal{L}(w_l, v_{l,C,f}) \cap \mathcal{L}(u_f, v_{l,1,f}) \subset Y_{l,f}$. $\diamondsuit$

By Lemma 4.7, if an lf-chain has exactly 3 interval sets $X_{l,f}$, $Y_{l,f}$ and $Z_{l,f}$, one of them, say $X_{l,f}$, will be a subset of $\mathcal{L}(u_f, v_{l,1,f}) - \mathcal{L}(w_l, v_{l,C,f})$, and another one of them, say $Z_{l,f}$, will be a subset of $\mathcal{L}(w_l, v_{l,C,f}) - \mathcal{L}(u_f, v_{l,1,f})$, and the last one $Y_{l,f}$ will contain the elements of $\mathcal{L}(u_f, v_{l,1,f}) \cap \mathcal{L}(w_l, v_{l,C,f})$, if any. Hereafter, we use $X_{l,f}$, $Y_{l,f}$ and $Z_{l,f}$ for the above meaning.

Lemma 4.8 $\forall l \in [1, L], f \in [1, F]$, *assume an lf-chain has exactly 3 interval sets. Let $A = \mathcal{L}(w_l, v_{l,C,f}) - X_{l,f} - Y_{l,f} - Z_{l,f}$ and $B = \mathcal{L}(u_f, v_{l,1,f}) - X_{l,f} - Y_{l,f} - Z_{l,f}$. If $X_{l,f} \prec Y_{l,f} \prec Z_{l,f} \prec X_{l,f}$, we have either*

$$\begin{array}{rl} & X_{l,f} \prec B \prec Y_{l,f} \prec A \prec Z_{l,f} \prec X_{l,f} \\ \textit{or} & X_{l,f} \prec B \prec Y_{l,f} \prec Z_{l,f} \prec A \prec X_{l,f} \\ \textit{or} & X_{l,f} \prec Y_{l,f} \prec A \prec Z_{l,f} \prec B \prec X_{l,f} \end{array}$$

If $Z_{l,f} \prec Y_{l,f} \prec X_{l,f} \prec Z_{l,f}$, we have either

$$\begin{array}{rl} & Z_{l,f} \prec A \prec Y_{l,f} \prec B \prec X_{l,f} \prec Z_{l,f} \\ \textit{or} & Z_{l,f} \prec A \prec Y_{l,f} \prec X_{l,f} \prec B \prec Z_{l,f} \\ \textit{or} & Z_{l,f} \prec Y_{l,f} \prec B \prec X_{l,f} \prec A \prec Z_{l,f} \end{array}$$

Proof: We will give the outline proof of the case $X_{l,f} \prec Y_{l,f} \prec Z_{l,f} \prec X_{l,f}$, and leave the other case to the reader.

Since none of $(X_{l,f} \cup Y_{l,f})$, $(Y_{l,f} \cup Z_{l,f})$ and $(X_{l,f} \cup Z_{l,f})$ is an interval, these 3 interval sets $X_{l,f}$, $Y_{l,f}$ and $Z_{l,f}$ divide the set $V_{L,C,F} - X_{l,f} - Y_{l,f} - Z_{l,f}$ into 3 intervals, say, P, Q and R. Without loss of generality, we have the cyclic structure $P \prec X_{l,f} \prec Q \prec Y_{l,f} \prec R \prec Z_{l,f} \prec P$.

Using the same technique as the proof of Lemma 4.5, we can prove that only one of P, Q and R can contain A and another one can contain B. The one containing A should be next to $Z_{l,f}$ and the one containing B should be next to $X_{l,f}$. Hence, the result follows. $\diamondsuit$

Lemma 4.9 *Given an $l \in [1, L]$ and an interval A containing $\{v_{l,C,f}|f \in [1, F']\}$, where $3 \leq F' \leq F$, and $b \prec v_{l,C,1} \prec \{v_{l,C,2}, v_{l,C,3}, \ldots, v_{l,C,F'-1}\} \prec v_{l,C,F'} \prec b$, where $b \notin A$. Assume the routing paths from w_l through $v_{l,C,f'}$ is shorter than $\frac{3}{2}D - 1$, $\forall v_{l,C,f'} \in A$ where $f' \in [1, F]$, and assume the lf-chains are normal, $\forall f \in [1, F']$. Then, we have at most 2 lf-chains, $f \in [2, F' - 1]$, each of which has exactly 3 interval sets.*

Proof: Assume there are at least 3 lf-chains, and each has exactly 3 interval sets. Without loss of generality, choose the first three, and let these 3 chains be $l2$-chain, $l3$-chain and $l4$-chain. Recall that for $f = 2, 3, 4$, $X_{l,f} \subset \mathcal{L}(u_f, v_{l,1,f}) - \mathcal{L}(w_l, v_{l,C,f})$ and $Z_{l,f} \subset \mathcal{L}(w_l, v_{l,C,f}) - \mathcal{L}(u_f, v_{l,1,f})$.

For $f = 2, 3, 4$, since $\mathcal{L}(w_l, v_{l,C,f})$ cannot contain $X_{l,f}$, if the cyclic order is $b_1 \prec v_{l,C,1} \prec X_{l,f} \prec v_{l,C,F'} \prec b_2 \prec b_1$, the assumption of path length from w_l ($\frac{3}{2}D-1$) will be violated, by using the same argument in the first point of the proof of Lemma 4.4. Hence, for $f = 2, 3, 4$, $X_{l,f} \prec v_{l,C,1} \prec \{v_{l,C,2}, \ldots, v_{l,C,F'-1}\} \prec v_{l,C,F'} \prec X_{l,f}$. Without loss of generality, assume $X_{l,2} \prec X_{l,3} \prec X_{l,4} \prec v_{l,C,1} \prec \{v_{l,C,2}, \ldots, v_{l,C,F'-1}\} \prec v_{l,C,F'} \prec X_{l,2}$, as Figure 9 in the proof of Lemma 4.6, where f_i in Figure 9 is considered to be $i + 1$, for $i = 1, 2, 3$.

Since $l3$-chain is normal, $\mathcal{L}(u_3, v_{l,1,3}) \cup \mathcal{L}(w_l, v_{l,C,3})$ is an interval and therefore, $\mathcal{L}(u_3, v_{l,1,3})$ will contain either $X_{l,2}$ or $X_{l,4}$, say $X_{l,2}$ as X_{l,f_1} in Figure 9. So, the routing path from u_3 to $X_{l,2}$ will be $u_3, w_l, u_{f'}, w_{l'}, u_2, X_{l,2}$, where $f' \neq 2, 3$ and $l' \neq l$. The length of this routing path is longer than $2D$ and contradicts to the assumption on the longest path. $\diamondsuit$

Lemma 4.10 *Given an $l \in [1, L]$ and an interval A containing $\{v_{l,C,f} | f \in [1, F']\}$, where $3 \le F' \le F$, and $b \prec v_{l,C,1} \prec \{v_{l,C,2}, v_{l,C,3}, \ldots, v_{l,C,F'-1}\} \prec v_{l,C,F'} \prec b$, where $b \notin A$. Assume the lf-chains are normal, $\forall f \in [1, F']$. Let $3 \le T \le F' - 2$. If there are T lf-chains, $f \in [2, F' - 1]$, having exactly three interval sets, we have at least $T - 2$ of these chains which are contained in A.*

Proof: Using the similar argument in the first part of the proof of Lemma 4.6. $\diamondsuit$

Lemma 4.11 *Given an $f \in [1, F]$ and an interval A containing $\{v_{l,1,f} | l \in [1, L]\}$, and $b \prec v_{1,1,f} \prec \{v_{2,1,f}, v_{3,1,f}, \ldots, v_{L-1,1,f}\} \prec v_{L,1,f} \prec b$, where $b \notin A$. Assume the lf-chains are normal, $\forall l \in [1, L]$. Let $3 \le T \le L - 2$. If there are T lf-chains, $l \in [2, L - 1]$, having exactly two or exactly three interval sets, we have at least $T - 2$ of these chains which are contained in A.*

Proof: Using the similar argument in the first part of the proof of Lemma 4.6. $\diamondsuit$

5 The Main Lemma

Lemma 5.1 *Assume there exists a labeling scheme such that every path is shorter than $2D - K$, $K \ge 3$. Given that there are 2 disjoint intervals A, B such that A contains $\{v_{l,C,f} | f \in [1, (2(L-2)(20(3\lfloor \frac{D}{K} \rfloor + 15) + 4) + 3)(L - 1) + 2]\} \cup \{v_{l_a,1,f} | f \in [1, (2(L - 2)(20(3\lfloor \frac{D}{K} \rfloor + 15) + 4) + 3)(L - 1) + 2]\}$, B contains $\{v_{l_b,1,f} | f \in [1, (2(L-2)(20(3\lfloor \frac{D}{K} \rfloor +$*

$15)+4)+3)(L-1)+2]\}$, *and* $v_{l_a,1,f} \in \mathcal{L}(w_l, v_{l,C,f})$, *where* $\forall f \in [1,(2(L-2)(20(3\lfloor\frac{D}{K}\rfloor + 15) + 4) + 3)(L-1) + 2]$. *Then,* $\exists f \in \{\pi \in [1,F] \mid v_{l,C,\pi} \in A\}$ *such that a routing path from* w_l *passing through the edge* $(w_l, v_{l,C,f})$ *no shorter than* $\frac{3}{2}D - 1$.

Intuitively, the purpose of this lemma is to show that if we allow some disjointness of layers, say l_a, l_bth layers, we can find a path from a w_l of length $\frac{3}{2}D - 1$. Such a path will not contradict our longest path assumption. However, after some argument in the proof of Theorem 6.1, we will find out that such a path of length $\frac{3}{2}D-1$ will come up with a path from a u_f of length $2D-1$. In the lemma statement, there are 3 layers—l, l_a and l_bth layers concerned. The reader will then find 5 layers—l, l_a, l_b, l', l''th layers in the proof. Among these 5 layers, l, l_a, l_bth layers are the platforms on which l, l' and l''th layers will play an important role in the proof.

Proof of the Main Lemma 5.1

Since this proof of main lemma is complicated, we use 14 Claims to make it simpler. A reader may also get enough details about this proof without looking into the Reasons of each claims.

Assume there exist a labeling scheme such that every path is shorter than $2D - K$, $K \geq 3$. And $\forall v_{l,C,f} \in A$, assume that all routing paths from w_l passing through $v_{l,C,f}$ is shorter than $\frac{3}{2}D - 1$. It is also important to point out the initial condition of this lemma as (1).

$$\forall f \in [1,(2(L-2)(20(3\lfloor\frac{D}{K}\rfloor + 15) + 4) + 3)(L-1) + 2], \quad v_{l_a,1,f} \in \mathcal{L}(w_l, v_{l,C,f}). \quad (1)$$

Since every interval has 2 margins, there are 2 elements $a_1, a_2 \in \{v_{l_a,1,f} | f \in [1,(2(L-2)(20(3\lfloor\frac{D}{K}\rfloor + 15) + 4) + 3)(L-1) + 2]\}$ such that $\exists$ an interval, which is not a subinterval of A, containing a_1, a_2 but not containing $\{v_{l_a,1,f} | f \in [1,(2(L-2)(20(3\lfloor\frac{D}{K}\rfloor + 15) + 4) + 3)(L-1) + 2]\} - \{a_1, a_2\}$. Without loss of generality, for $i = 1, 2$, we assume a_i to be $v_{l_a,1,f_i}$, and $f_i = (2(L-2)(20(3\lfloor\frac{D}{K}\rfloor + 15) + 4) + 3)(L-1) + i$. Therefore, we have the cyclic structure $a_1 \prec \{v_{l_a,1,f} | f \in [1,(2(L-2)(20(3\lfloor\frac{D}{K}\rfloor + 15) + 4) + 3)(L-1)]\} \prec a_2 \prec B \prec a_1$. (Figure 10)

Claim 1 *The routing path from* w_l *to any element in* $\langle a_1, a_2\rangle - \{a_1, a_2\}$ *should be shorter than* $\frac{3}{2}D - 1$.

Reason: $\forall x \in \langle a_1, a_2\rangle - \{a_1, a_2\}$, assume the routing from w_l to x will start with edge $(w_l, v_{l,C,f})$. We have two cases.

- $v_{l,C,f} \in A$.
  By the assumption on the routing path from w_l, this routing path should be shorter than $\frac{3}{2}D - 1$.

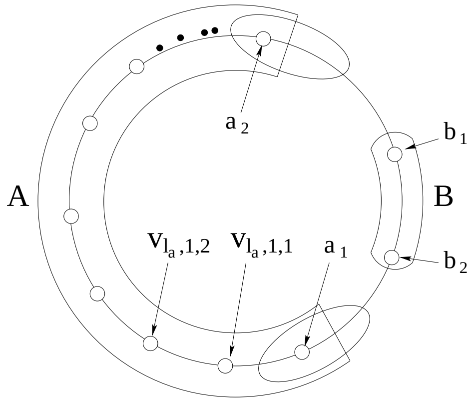

Figure 10: Cyclic structure of A and B (Not to scale).

- $v_{l,C,f} \notin A$.
  $x \in \mathcal{L}(w_l, v_{l,C,f}) \Rightarrow a_i \in \mathcal{L}(w_l, v_{l,C,f})$, $i = 1$ or 2, It is a contradiction to (1), or a contradiction to Property 2, because for $i = 1$ or 2, $a_i \in \mathcal{L}(w_l, v_{l,C,f_i})$ where $v_{l,C,f_i} \in A$.

♠

Claim 2 *The routing from u_f, $\forall f \in [1, F]$, to any elements in $\{v_{l_b,1,f} | f \in [1, (2(L-2)(20(3\lfloor \frac{D}{K} \rfloor + 15) + 4) + 3)(L-1)]\}$ cannot pass through w_l.*

Reason: Assume the contrary and the routing from u_f, $f \in [1, F]$, to an element $v_{l_b,1,f'}$, $f' \in [1, (2(L-2)(20(3\lfloor \frac{D}{K} \rfloor + 15) + 4) + 3)(L-1)]$, passes through w_l. If $f = f'$, it is a contradiction because $\mathcal{L}(u_f, v_{l_b,1,f})$ must contain $v_{l_b,1,f'}$; otherwise, the longest path assumption $(2D - K)$ is violated. If $f \neq f'$, $\mathcal{L}(w_l, v_{l,C,f'})$ must contain $v_{l_b,1,f'}$; otherwise, the routing path from u_f to $v_{l_b,1,f'}$ will be $u_f, w_l, u_{f''}, w_{l_b}, v_{l_b,1,f'}$, where $f'' \neq f, f'$, and of length $2D - 1$. Hence, $\mathcal{L}(w_l, v_{l,C,f'})$ contains $v_{l_b,1,f'}$, and also contains $v_{l_a,1,f'}$, according to (1). However, $v_{l_a,1,f'}$ and $v_{l_b,1,f'}$ are in two disjoint intervals—A and B, respectively, and therefore, $\mathcal{L}(w_l, v_{l,C,f'})$ will contain one of the marginal elements of A, say a_1. Contradiction of (1) or Property 2 on w_l. ♠

By Claim 2, the routing from $u_f, \forall f \notin [1, (2(L-2)(20(3\lfloor \frac{D}{K} \rfloor + 15) + 4) + 3)(L-1)]$, to $\{v_{l_b,1,f} | f \in [1, (2(L-2)(20(3\lfloor \frac{D}{K} \rfloor + 15) + 4) + 3)(L-1)]\}$ will not pass through w_l. Then, without w_l, there are still $L - 1$ choices. By the Pigeon Hole Principle, there are $2(L-2)(20(3\lfloor \frac{D}{K} \rfloor + 15) + 4) + 3$ elements in $\{v_{l_b,1,f} | f \in [1, (2(L-2)(20(3\lfloor \frac{D}{K} \rfloor + 15) + 4) + 3)(L-1)]\}$ to which the routing path from u_f will pass through $w_{l'}$, $l' \neq l$. Without loss of generality, we assume these $2(L-2)(20(3\lfloor \frac{D}{K} \rfloor + 15) + 4) + 3$ elements are belonged to the first $2(L-2)(20(3\lfloor\frac{D}{K}\rfloor+15)+4)+3$ flaps. Therefore,

$$\forall f \in [1, 2(L-2)(20(3\lfloor\frac{D}{K}\rfloor+15)+4)+3], \quad v_{l_b,1,f} \in \mathcal{L}(w_{l'}, v_{l',C,f}). \tag{2}$$

By the same argument in the second paragraph in page 16, there are 2 elements b_1, b_2 such that $b_1 \prec \{v_{l_b,1,f} | f \in [1, 2(L-2)(20(3\lfloor\frac{D}{K}\rfloor+15)+4)+1]\} \prec b_2 \prec A \prec b_1$ (Figure 10), where for $j=1,2$, b_j is $v_{l_b,1,f_j}$, $f_j = 2(L-2)(20(3\lfloor\frac{D}{K}\rfloor+15)+4)+1+j$.

Let A' be an interval containing $\{v_{l,C,f} | f \in [1, 2(L-2)(20(3\lfloor\frac{D}{K}\rfloor+15)+4)+1]\} \cup \{v_{l_a,1,f} | f \in [1, 2(L-2)(20(3\lfloor\frac{D}{K}\rfloor+15)+4)+1]\}$ and B' contains $\{v_{l_b,1,f} | f \in [1, 2(L-2)(20(3\lfloor\frac{D}{K}\rfloor+15)+4)+1]\}$. Obviously, $A' \subset A$, $B' \subset B$. Let $p = 2(L-2)(20(3\lfloor\frac{D}{K}\rfloor+15)+4)+1$. Without loss of generality, assume the cyclic order in A is

$$a_2 \prec B \prec a_1 \prec \{v_{l_a,1,1}, \dots, v_{l_a,1,(L-2)(20(3\lfloor\frac{D}{K}\rfloor+15)+4)}\} \prec \underline{v_{l_a,1,p}} \prec$$
$$\{v_{l_a,1,(L-2)(20(3\lfloor\frac{D}{K}\rfloor+15)+4)+1}, \dots, v_{l_a,1,2(L-2)(20(3\lfloor\frac{D}{K}\rfloor+15)+4)}\} \prec a_2$$

Note that $v_{l_a,1,p}$ is the "middle" element in A' if we only consider the l_ath layer in A'. Also, note that there may be some $v_{l_a,1,f}$'s not belonged to $\{v_{l_a,1,f} | f \in [1, 2(L-2)(20(3\lfloor\frac{D}{K}\rfloor+15)+4)+1]\}$ but in A'.

Consider $v_{l,2,p}$. $\mathcal{L}(v_{l,2,p}, v_{l,1,p})$ should contain $v_{l_a,1,p}$ and $v_{l_b,1,p}$; otherwise the routing path from $v_{l,2,p}$ to $v_{l_a,1,p}$ or to $v_{l_b,1,p}$ is $2D-3$. Hence, $\mathcal{L}(v_{l,2,p}, v_{l,1,p})$ should contain some elements of A and b_i, where $i=1$ or 2 (interval H in figure 11). Without loss of generality, assume that $v_{l_a,1,1}, v_{l_a,1,2}, \dots, v_{l_a,1,(L-2)(20(3\lfloor\frac{D}{K}\rfloor+15)+4)} \in \mathcal{L}(v_{l,2,p}, v_{l,1,p})$.

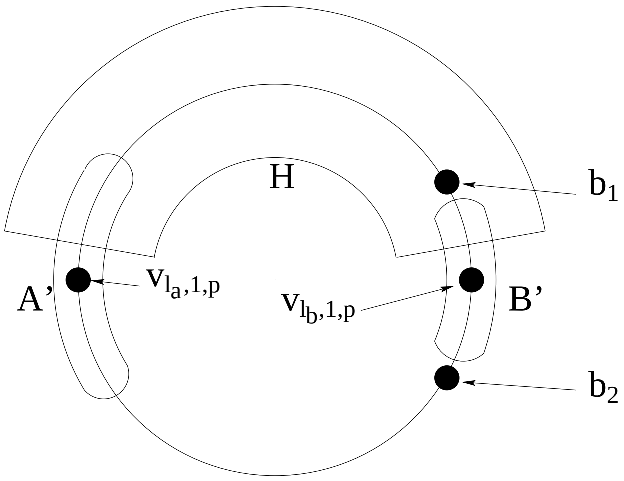

Figure 11: The structure of A', B' and H.

Claim 3 *The routing from $v_{l,2,p}$ to any element in $\{v_{l_a,1,1}, v_{l_a,1,2}, \dots, v_{l_a,1,(L-2)(20(3\lfloor\frac{D}{K}\rfloor+15)+4)}\}$ cannot pass through $v_{l',1,p}$.*

Reason: Assume a routing path from $v_{l,2,p}$ to $v_{l_a,1,i}$ passing through $v_{l',1,p}$, $i \in [1,(L-2)(20(3\lfloor\frac{D}{K}\rfloor+15)+4)]$. Then, $\mathcal{L}(w_{l'}, v_{l',C,i})$ will contain $v_{l_a,1,i}$; otherwise, the path from $v_{l,2,p}$ to $v_{l_a,1,i}$ will have length $2D+1$. Recall that $\mathcal{L}(w_{l'}, v_{l',C,i})$ contains $v_{l_b,1,i}$. Then, $\mathcal{L}(w_{l'}, v_{l',C,i})$ contains $v_{l_a,1,i}$ and $v_{l_b,1,i}$, implying that $\mathcal{L}(w_{l'}, v_{l',C,i})$ will either contain b_1 or b_2, say b_1 (Figure 11). Recall that $b_1 = v_{l_b,1,f} \in \mathcal{L}(w_{l'}, v_{l',C,f})$, where $f = 2(L-2)(20(3\lfloor\frac{D}{K}\rfloor+15)+4)+2$. Hence, Property 2 on $w_{l'}$ is violated. ♠

By Claim 3, the routing from $v_{l,2,p}$ to any element in $\{v_{l_a,1,f} | f \in [1,(L-2)(20(3\lfloor\frac{D}{K}\rfloor+15)+4)]\}$ should pass through other layers. Other than lth and l'th layers, we have $L-2$ remaining layers. By the Pigeon hole Principle, $\exists$ a layer, say l''th layer, through which the routing from $v_{l,2,p}$ to at least $20(3\lfloor\frac{D}{K}\rfloor+15)+4$ elements in $\{v_{l_a,1,f} | f \in [1,(L-2)(20(3\lfloor\frac{D}{K}\rfloor+15)+4)]\}$ will pass. Without loss of generality, we assume that the routing from $v_{l,2,p}$ to $\{v_{l_a,1,f} | f \in [1, 20(3\lfloor\frac{D}{K}\rfloor+15)+4]\}$ will pass through the l''th layer. Assume $a_1 \prec v_{l_a,1,20(3\lfloor\frac{D}{K}\rfloor+15)+4} \prec v_{l_a,1,1} \prec v_{l_a,1,2} \prec \cdots \prec v_{l_a,1,20(3\lfloor\frac{D}{K}\rfloor+15)+2} \prec v_{l_a,1,20(3\lfloor\frac{D}{K}\rfloor+15)+3} \prec a_2 \prec B \prec a_1$. Hence, we have

$$\forall x, \quad v_{l_a,1,20(3\lfloor\frac{D}{K}\rfloor+15)+4} \prec x \prec v_{l_a,1,20(3\lfloor\frac{D}{K}\rfloor+15)+3} \Rightarrow x \in \mathcal{L}(u_p, v_{l'',1,p}). \tag{3}$$

Obviously, under the assumption of longest path $(2D-K)$,

$$\forall f \in [1, 20(3\lfloor\frac{D}{K}\rfloor+15)+2], \quad v_{l_a,1,f} \in \mathcal{L}(w_{l''}, v_{l'',C,f}); \tag{4}$$

otherwise, the routing path from u_p to $v_{l_a,1,f}$, $f \in [1, 20(3\lfloor\frac{D}{K}\rfloor+15)+8]$, will be longer than $2D-3$. By Property 2 on $w_{l''}$, we have

$$\begin{aligned} a_1 \prec \quad & v_{l_a,1,20(3\lfloor\frac{D}{K}\rfloor+15)+4} \prec \mathcal{L}(w_{l''}, v_{l'',C,1}) \prec \mathcal{L}(w_{l''}, v_{l'',C,2}) \prec \cdots \prec \\ & \mathcal{L}(w_{l''}, v_{l'',C,20(3\lfloor\frac{D}{K}\rfloor+15)+2}) \prec v_{l_a,1,20(3\lfloor\frac{D}{K}\rfloor+15)+3} \prec a_2 \prec B \prec a_1. \end{aligned} \tag{5}$$

Hence, $\forall f \in [1, 20(3\lfloor\frac{D}{K}\rfloor+15)+2]$, $\mathcal{L}(w_{l''}, v_{l'',C,f}) \subset \mathcal{L}(u_p, v_{l'',1.p})$. Therefore, all routing paths from $w_{l''}$ pass through $v_{l'',C,f}$, which is belonged to $\langle v_{l'',C,1}, v_{l'',C,20(3\lfloor\frac{D}{K}\rfloor+15)+2}\rangle$, are shorter than $\frac{3}{2}D-1$; otherwise at least one routing path from u_p through $w_{l''}$ will be longer than $2D-3$. Also, since $\mathcal{L}(w_{l''}, v_{l'',C,f})$ contains $v_{l'',C,f}$, $\forall f \in [1, 20(3\lfloor\frac{D}{K}\rfloor+15)+2]$, by (5), we have

$$\begin{aligned} & a_1 \prec v_{l_a,1,20(3\lfloor\frac{D}{K}\rfloor+15)+4} \prec \{v_{l'',C,1}, v_{l_a,1,1}\} \prec \{v_{l'',C,2}, v_{l_a,1,2}\} \prec \cdots \prec \\ & \{v_{l'',C,20(3\lfloor\frac{D}{K}\rfloor+15)+2}, v_{l_a,1,20(3\lfloor\frac{D}{K}\rfloor+15)+2}\} \prec v_{l_a,1,20(3\lfloor\frac{D}{K}\rfloor+15)+3} \prec a_2 \prec B \prec a_1. \end{aligned} \tag{6}$$

Claim 4 *The routing path from $w_{l''}$ to any element in $\langle v_{l'',C,1}, v_{l'',C,20(3\lfloor\frac{D}{K}\rfloor+15)+2}\rangle$ should be shorter than $\frac{3}{2}D-1$.*

Reason: Similar technique as the reason of Claim 1. ♠

Now we are going to focus on the l_ath, lth, l'th and the l''th layer in A. (Note that we allow A to contain some flaps of the l'th layer. Of course, some are in B).

Ignore the 1st and the $20(3\lfloor\frac{D}{K}\rfloor+15)+2$th flaps, (as ignored by Lemma 4.6 and 4.9), we have $20(3\lfloor\frac{D}{K}\rfloor+15)$ flaps left. By Lemma 4.2, the lth, l'th and the l''th layers totally have at most $3\lfloor\frac{D}{K}\rfloor+6$ abnormal chains. Among those normal chains, by Lemma 4.6, there are at most 4 chains, 2 from the lth and 2 from the l'' layer, having exactly 2 interval sets, which is not totally included in $\langle v_{l'',C,1}, v_{l'',C,20(3\lfloor\frac{D}{K}\rfloor+15)+2}\rangle$, the routing from w_l or $w_{l''}$ to these chains will need at least $\frac{3}{2}D-1$. Also, among those normal chains, by Lemma 4.9, there are at most 4 chains, 2 from the lth and 2 from the l'' layer, having exactly 3 interval sets.

Summing up, there are $3\lfloor\frac{D}{K}\rfloor+14$ chains which are not of our interest. By the Pigeon Hole Principle, we can find an interval A'', which is a subinterval of $\langle v_{l'',C,1}, v_{l'',C,20(3\lfloor\frac{D}{K}\rfloor+15)+2}\rangle$, containing no elements $v_{l,C,f}, v_{l',C,f}, v_{l'',C,f}$ from abnormal chains from the lth, l'th, l''th layers, respectively; no elements $v_{l,C,f}, v_{l'',C,f}$ of normal chains from the lth and l''th layer, respectively, having exactly 3 interval sets; no elements $v_{l,C,f}, v_{l'',C,f}$ of normal chains from the lth and l''th layer, respectively, having exactly 2 interval sets with some elements in the chains not in $\langle v_{l'',C,1}, v_{l'',C,20(3\lfloor\frac{D}{K}\rfloor+15)+2}\rangle$, $f\in[1,F]$.

The interval A'' contains elements of the lth, l'th and the l''th layers from 19 flaps out of the said $20(3\lfloor\frac{D}{K}\rfloor+15)$ flaps. Let these 19 flaps be the f_ith flaps (Here, we redefine f_i), $i=1,\ldots,19$. Without loss of generality, we assume

$$a_1 \prec v_{l_a,1,f_1} \prec v_{l_a,1,f_2} \prec \cdots \prec v_{l_a,1,f_{19}} \prec a_2 \prec B \prec a_1.$$

By (1) and Property 2 on w_l, we have

$$a_1 \prec \mathcal{L}(w_l, v_{l,C,f_1}) \prec \mathcal{L}(w_l, v_{l,C,f_2}) \prec \cdots \prec \mathcal{L}(w_l, v_{l,C,f_{19}}) \prec a_2 \prec B \prec a_1. \tag{7}$$

Combining (7) with (5) and (6), we have

$$\begin{aligned} a_1 \prec\ & \mathcal{L}(w_l, v_{l,C,f_1}) \cap \mathcal{L}(w_{l''}, v_{l'',C,f_1}) \prec \mathcal{L}(w_l, v_{l,C,f_2}) \cap \mathcal{L}(w_{l''}, v_{l'',C,f_2}) \\ & \prec \cdots \prec \mathcal{L}(w_l, v_{l,C,f_{19}}) \cap \mathcal{L}(w_{l''}, v_{l'',C,f_{19}}) \prec a_2 \prec B \prec a_1. \end{aligned} \tag{8}$$

Claim 5 *For* $i=1,\ldots,19$, $\{v_{l,1,f_i}, v_{l'',1,f_i}\} \subset \mathcal{L}(w_l, v_{l,C,f_i}) \cap \mathcal{L}(w_{l''}, v_{l'',C,f_i})$.

Reason: From our choice of these 19 flaps, $v_{l,1,f_i}, v_{l'',1,f_i} \in \langle v_{l'',C,1}, v_{l'',C,20(3\lfloor\frac{D}{K}\rfloor+15)+2}\rangle$, $\forall i\in[1,19]$. By Claim 1 and 4, the routing path from any one of $w_l, w_{l''}$ to any one of $v_{l,1,f_i}, v_{l'',1,f_i}$ should be shorter than $\frac{3}{2}D-1$. Hence, the Claim statement should hold, otherwise, the routing path should be at least $\frac{3}{2}D+1$. ♠

By Claim 5 and (8),

$$a_1 \prec \{v_{l,1,f_1}, v_{l_a,1,f_1}, v_{l'',1,f_1}\} \prec \cdots \prec \{v_{l,1,f_{19}}, v_{l_a,1,f_{19}}, v_{l'',1,f_{19}}\} \prec a_2 \prec B \prec a_1 \tag{9}$$

On the other hand, the cyclic structure inside B is

$$b_1 \prec v_{l_b,1,\sigma_1} \prec \cdots \prec v_{l_b,1,\sigma_{19}} \prec b_2$$

where $\sigma_1\sigma_2\ldots\sigma_{19}$ is a certain permutation of $f_1 f_2 \ldots f_{19}$. Without loss of generality, we assume $\{f_1,\ldots,f_{17}\} \cap \{\sigma_1,\sigma_{19}\} = \emptyset$. (Otherwise, instead of using f_i's, we can use another subscript g_i's such that $\{g_1,\ldots,g_{17}\} \cap \{\sigma_1,\sigma_{19}\} = \emptyset$).

Claim 6 *The routing from $w_{l'}$ to any element of $\{v_{l'',1,f_1}, v_{l'',1,f_2},\ldots,v_{l'',1,f_{17}}\}$, cannot pass through any edges among $(w_{l'}, v_{l',C,f_1})$, $(w_{l'}, v_{l',C,f_2})$, $\ldots$, $(w_{l'}, v_{l',C,f_{17}})$.*

Reason: For $i = 1,\ldots,17$, if $\mathcal{L}(w_{l'}, v_{l',C,f_i})$ contains $v_{l'',1,f_i}$, $\mathcal{L}(w_{l'}, v_{l',C,f_i})$ will contain at least one of $v_{l_b,1,\sigma_1}, v_{l_b,1,\sigma_{19}}$, where $f_i \neq \sigma_1, \sigma_{19}$ (Figure 12). By (2), for

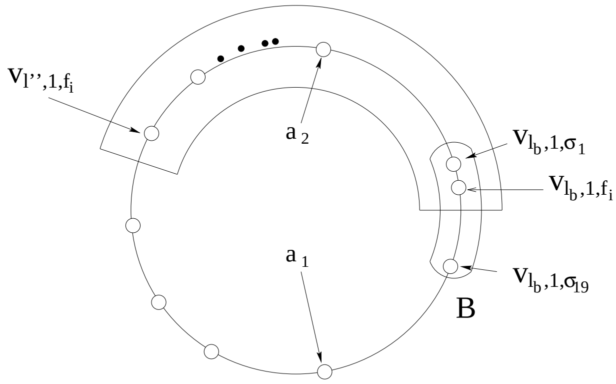

Figure 12: At least one of $v_{l_b,1,\sigma_1}, v_{l_b,1,\sigma_{11}}$ is in $\mathcal{L}(w_{l'}, v_{l',C,f_i})$.

$j = 1, 19$, $v_{l_b,1,\sigma_j} \in \mathcal{L}(w_{l'}, v_{l',C,\sigma_j})$. In other words, $\exists i \in \{1,\ldots,17\}, j \in \{1,19\}$, $\mathcal{L}(w_{l'}, v_{l',C,f_i}) \cap \mathcal{L}(w_{l'}, v_{l',C,\sigma_j}) \neq \emptyset$, where $f_i \neq \sigma_j$. Contradiction to Property 2 on $w_{l'}$. ♠

Claim 7 *The routing from $w_{l'}$ to any 5 elements of $\{v_{l'',1,f_1},\ldots,v_{l'',1,f_{17}}\}$, cannot pass through only an edge $(w_{l'}, v_{l',C,\alpha})$, $\alpha \neq f_1,\ldots,f_{17}$.*

Reason: Assume $\mathcal{L}(w_{l'}, v_{l',C,\alpha})$ contains 5 elements of $\{v_{l'',1,f_1},\ldots,v_{l'',1,f_{17}}\}$, say $v_{l'',1,f_1}, v_{l'',1,f_2},\ldots,v_{l'',1,f_5}$. Since, by Property 2 on $w_{l'}$, $\mathcal{L}(w_{l'}, v_{l',C,\alpha})$ cannot contain any element in $\{v_{l_b,1,\sigma_1}, v_{l_b,1,\sigma_{19}}\}$, $\mathcal{L}(w_{l'}, v_{l',C,\alpha})$ contains $\langle v_{l'',1,f_1}, v_{l'',1,f_5}\rangle$. By (8), we have

$$v_{l'',1,f_1} \prec v_{l'',C,f_2} \prec v_{l'',C,f_3} \prec v_{l'',C,f_4} \prec v_{l'',1,f_5}$$

and

$$v_{l'',1,f_1} \prec v_{l,C,f_2} \prec v_{l,C,f_3} \prec v_{l,C,f_4} \prec v_{l'',1,f_5}$$

Then, at least one of the following is true.

- $v_{l'',C,f_2} \prec \{v_{l'',C,f_3}, v_{l,C,f_i}\} \prec v_{l'',C,f_4}$
- $v_{l,C,f_2} \prec \{v_{l,C,f_3}, v_{l'',C,f_i}\} \prec v_{l,C,f_4}$,

where $i = 2$ or 4.

Without loss of generality, we assume the first case. Consider the routing from $w_{l'}$ to v_{l'',C,f_2}, v_{l'',C,f_3} and v_{l'',C,f_4}. The routing path will be $w_{l'}, u_\alpha, w_{l''}, v_{l'',C,f_i}$, $i = 2, 3, 4$. Then, $\langle v_{l'',C,f_2}, v_{l'',C,f_4}\rangle \subset \mathcal{L}(u_\alpha, v_{l'',1,\alpha})$; otherwise, $B \subset \mathcal{L}(u_\alpha, v_{l'',1,\alpha})$ (Figure 13), which implies a routing path—$u_\alpha, w_{l''}, u_{\neq\sigma_1}, w_{l_b}, v_{l_b,1,\sigma_1}$ of length $2D-1$

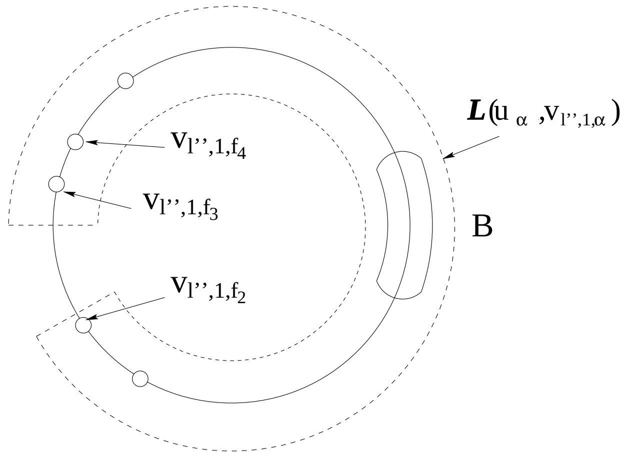

Figure 13: The case $\langle v_{l'',C,f_2}, v_{l'',C,f_4}\rangle \not\subset \mathcal{L}(u_\alpha, v_{l'',1,\alpha})$.

from u_α to $v_{l_b,1,\sigma_1}$ ($\in B$) because $\mathcal{L}(w_{l''}, v_{l'',C,\sigma_1}) \cap B = \emptyset$. However, $\langle v_{l'',C,f_2}, v_{l'',C,f_4}\rangle \subset \mathcal{L}(u_\alpha, v_{l'',1,\alpha})$ implies $v_{l,C,f_i} \in \mathcal{L}(u_\alpha, v_{l'',1,\alpha})$, $i \in \{2, 4\}$. The routing path from $w_{l'}$ to v_{l,C,f_i} will be $w_{l'}, u_\alpha, w_{l''}, v_{l,C,f_i}$, which length is not less than $2D-1$. Contradiction to our assumption on longest path. ♠

Claim 8 *The routing from $w_{l'}$ to any 5 elements of $\{v_{l,1,f_1}, \ldots, v_{l,1,f_{17}}\}$, cannot pass through only an edge $(w_{l'}, v_{l',C,\alpha})$ only, $\alpha \neq f_1, \ldots, f_{17}$.*

Reason: Similar to the proof of Claim 7. ♠

From Claim 6 and 7, the routing from $w_{l'}$ to $\{v_{l'',1,f_1}, \ldots, v_{l'',1,f_{17}}\}$ will pass at least 5 (at most 17) edges of $w_{l'}$ which are not any of $(w_{l'}, v_{l',1,f_1})$,..., $(w_{l'}, v_{l',1,f_{17}})$. Some cases using 5 edges of $w_{l'}$ are shown in figure 14. No mather how many edges of $w_{l'}$ used for these routing, the routing from $w_{l'}$ to $v_{l'',1,f_1}$, $v_{l'',1,f_5}$, $v_{l'',1,f_9}$, $v_{l'',1,f_{13}}$ and $v_{l'',1,f_{17}}$ will use different edges of $w_{l'}$. Let the routing from $w_{l'}$ to $v_{l'',1,f_9}$ pass through the edge $(w_{l'}, v_{l',C,\alpha})$, hence, by Property 2 on $w_{l'}$, $\mathcal{L}(w_{l'}, v_{l',C,\alpha})$ is "bounded" by $w_{l'}$'s 2 interval labels containing $v_{l'',1,f_5}$ and $v_{l'',1,f_{13}}$, respectively. In

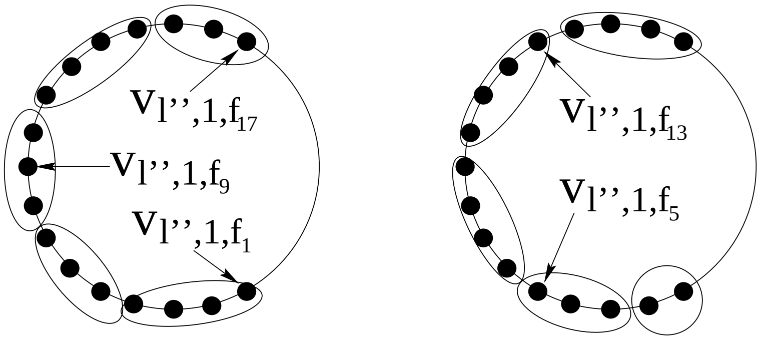

Figure 14: Some cases of using 5 edges of $w_{l'}$.

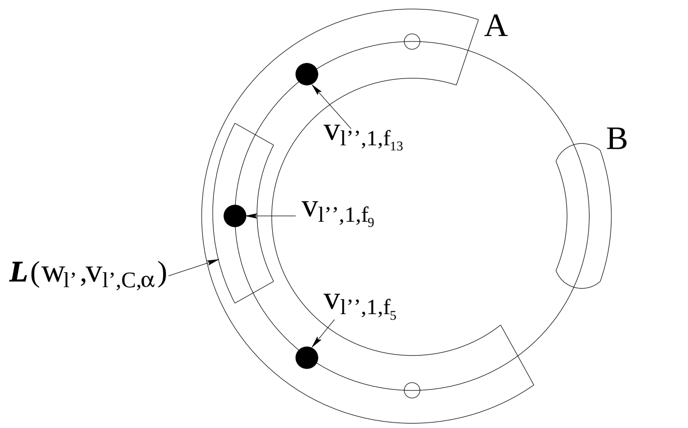

Figure 15: $\mathcal{L}(w_{l'}, v_{l',C,\alpha})$ is bounded by $v_{l'',1,f_5}$ and $v_{l'',1,f_{13}}$.

other words, $\mathcal{L}(w_{l'}, v_{l',C,\alpha})$ is "bounded" by $v_{l'',1,f_5}$ and $v_{l'',1,f_{13}}$ (Figure 15). Since $v_{l',C,\alpha} \in \mathcal{L}(w_{l'}, v_{l',C,\alpha})$, $v_{l',C,\alpha}$ is bounded by $v_{l'',1,f_5}$ and $v_{l'',1,f_{13}}$. We are now looking for a position for $v_{l',1,\alpha}$.

Recall that if $v_{l,C,f}, v_{l',C,f}$ and $v_{l'',C,f}$ are in A'', the lf-, $l'f$- and $l''f$-chains here are normal, respectively, where $f \in [1, F]$. Hence, $l'\alpha$-chain is normal. If $v_{l',1,\alpha} \notin \langle v_{l'',1,f_5}, v_{l'',1,f_{13}} \rangle$ (Figure 16), $\mathcal{L}(u_\alpha, v_{l',1,\alpha})$ will contain either $v_{l'',1,f_5}$ or $v_{l'',1,f_{13}}$, say

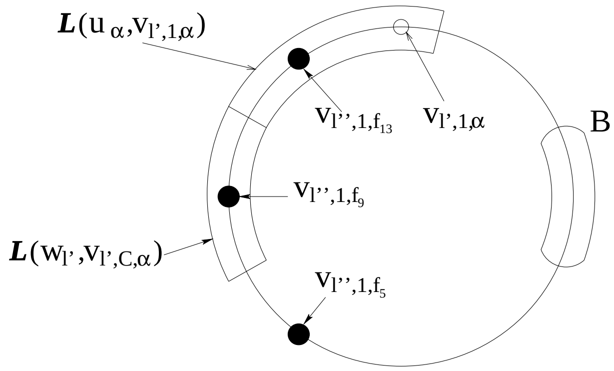

Figure 16: $\mathcal{L}(w_{l'}, v_{l',C,\alpha}) \cup \mathcal{L}(u_\alpha, v_{l',1,\alpha})$ is an interval.

$v_{l'',1,f_{13}}$ as Figure 16. So, the routing path from u_α to $v_{l'',1,f_{13}}$ is no shorter than $2D - 1$ since $v_{l'',1,f_{13}} \notin \mathcal{L}(w_{l'}, v_{l',C,f_{13}})$ by Claim 6. Hence, $v_{l',1,\alpha} \in \langle v_{l'',1,f_5}, v_{l'',1,f_{13}} \rangle$, ie. "bounded" by $v_{l,1,f_5}$ and $v_{l,1,f_{13}}$. So, there are 2 choices for the position of $v_{l',1,\alpha}$. One is $\langle v_{l'',1,f_5}, v_{l'',1,f_9} \rangle$ and the other is $\langle v_{l'',1,f_9}, v_{l'',1,f_{13}} \rangle$. Without loss of generality, we assume the former—"bounded" by $v_{l,1,f_5}$ and $v_{l,1,f_9}$. Now we will consider the routing from $w_l, w_{l''}$ to $v_{l',1,\alpha}$.

Since $a_1 \prec v_{l',1,\alpha} \prec a_2 \prec B \prec a_1$, by Claim 1, $v_{l',1,\alpha} \in \mathcal{L}(w_l, v_{l,C,\alpha})$. Since $v_{l',1,\alpha} \in \langle v_{l'',C,1}, v_{l'',C,20(3\lfloor \frac{D}{K} \rfloor + 15)+2} \rangle$, by Claim 4, $v_{l',1,\alpha} \in \mathcal{L}(w_{l''}, v_{l'',C,\alpha})$. Hence, $v_{l',1,\alpha} \in \mathcal{L}(w_l, v_{l,C,\alpha}) \cap \mathcal{L}(w_{l''}, v_{l'',C,\alpha})$. Also, by Property 2 on w_l and $w_{l''}$, we have

$$v_{l'',1,f_5} \prec \mathcal{L}(w_l, v_{l,C,\alpha}) \cap \mathcal{L}(w_{l''}, v_{l'',C,\alpha}) \prec v_{l'',1,f_9} \tag{10}$$

because $v_{l'',1,f_5} \in \mathcal{L}(w_l, v_{l,C,f_5}) \cap \mathcal{L}(w_{l''}, v_{l'',C,f_5})$ and $v_{l'',1,f_9} \in \mathcal{L}(w_l, v_{l,C,f_9}) \cap \mathcal{L}(w_{l''}, v_{l'',C,f_9})$.

By Lemma 4.6, $v_{l,1,\alpha}$ and $v_{l'',1,\alpha}$ are in A''. By Claim 1 and Claim 4, $v_{l,1,\alpha} \in \mathcal{L}(w_l, v_{l,C,\alpha})$ and $v_{l'',1,\alpha} \in \mathcal{L}(w_{l''}, v_{l'',C,\alpha})$. Again, by Claim 1 and Claim 4, $v_{l,1,\alpha} \in \mathcal{L}(w_{l''}, v_{l'',C,\alpha})$ and $v_{l'',1,\alpha} \in \mathcal{L}(w_l, v_{l,C,\alpha})$, because $a_1 \prec \{v_{l,1,\alpha}, v_{l'',1,\alpha}\} \prec a_2 \prec B \prec a_1$ and $\{v_{l,1,\alpha}, v_{l'',1,\alpha}\} \subset \langle v_{l'',C,1}, v_{l'',C,20(3\lfloor \frac{D}{K} \rfloor + 15)+2} \rangle$. Therefore,

$$\begin{gathered} \{v_{l,1,\alpha}, v_{l',1,\alpha}, v_{l'',1,\alpha}\} \subset \mathcal{L}(w_l, v_{l,C,\alpha}) \cap \mathcal{L}(w_{l''}, v_{l'',C,\alpha}) \\ v_{l'',1,f_5} \prec \{v_{l,1,\alpha}, v_{l',1,\alpha}, v_{l'',1,\alpha}\} \prec v_{l'',1,f_9} \end{gathered} \tag{11}$$

Claim 9 *The cyclic structure in (11) is either*

$v_{l'',1,f_5} \prec v_{l,1,\alpha} \prec v_{l',1,\alpha} \prec v_{l'',1,\alpha} \prec v_{l'',1,f_9}$ *or* $v_{l'',1,f_5} \prec v_{l'',1,\alpha} \prec v_{l',1,\alpha} \prec v_{l,1,\alpha} \prec v_{l'',1,f_9}$.

Reason: Assume the contrary, then we have 2 cases.

1 $v_{l'',1,f_5} \prec \{v_{l,1,\alpha}, v_{l'',1,\alpha}\} \prec v_{l',1,\alpha} \prec v_{l'',1,f_9}$.
Again, we have 2 further symmetric cases.

1.1 $v_{l'',1,f_5} \prec v_{l,1,\alpha} \prec v_{l'',1,\alpha} \prec v_{l',1,\alpha} \prec v_{l'',1,f_9}$.
Here, we have 2 different cases.

1.1.1 $l''\alpha$-chain has exactly one interval set.
Consider the $l''\alpha$-chain. Then, we have either $v_{l'',1,f_5} \prec v_{l,1,\alpha} \prec v_{l'',C,\alpha} \prec v_{l'',1,\alpha} \prec v_{l',1,\alpha} \prec v_{l'',1,f_9}$ or $v_{l'',1,f_5} \prec v_{l,1,\alpha} \prec v_{l'',1,\alpha} \prec v_{l'',C,\alpha} \prec v_{l',1,\alpha} \prec v_{l'',1,f_9}$. By Lemma 4.4, in the former case, $\mathcal{L}(w_{l''}, v_{l'',C,\alpha})$ will contain $v_{l',1,\alpha}$, but not $v_{l,1,\alpha}$; in the latter case, $\mathcal{L}(w_{l''}, v_{l'',C,\alpha})$ will contain $v_{l,1,\alpha}$, but not $v_{l',1,\alpha}$. Contradiction to (11).

1.1.2 $l''\alpha$-chain has exactly two interval sets—$X_{l'',\alpha}, Y_{l'',\alpha}$.
Consider the $l''\alpha$-chain. Assume $v_{l'',1,\alpha} \in X_{l'',\alpha}$, we have $v_{l'',1,f_5} \prec v_{l,1,\alpha} \prec X_{l'',\alpha} \prec v_{l',1,\alpha} \prec v_{l'',1,f_9}$. Then, by Lemma 4.5, we have $v_{l'',1,f_5} \prec v_{l,1,\alpha} \prec X_{l'',\alpha} \cup Y_{l'',\alpha} \prec v_{l',1,\alpha} \prec v_{l'',1,f_9}$, because $\mathcal{L}(w_{l''}, v_{l'',C,\alpha})$ cannot contain both $v_{l,1,\alpha}$ and $v_{l',1,\alpha}$ if $Y_{l'',\alpha}$ is in another place in the cyclic structure.
However, if $\mathcal{L}(w_{l''}, v_{l'',C,\alpha})$ contains $v_{l,1,\alpha}$ and $v_{l',1,\alpha}$ and $v_{l'',1,\alpha}$ (Figure 17), it will contain $v_{l'',1,f_9}$ which is belonged to $\mathcal{L}(w_{l''}, v_{l'',C,f_9})$.

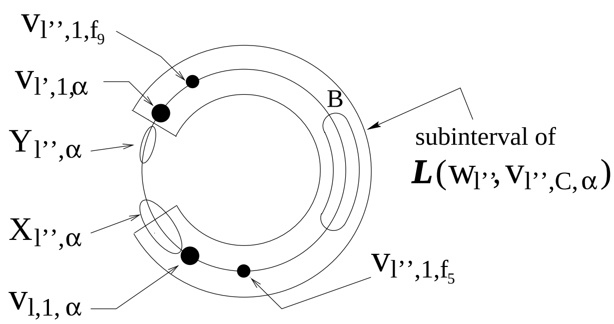

Figure 17: False example of cyclic structure.

Contradiction to Property 2 on $w_{l''}$.

1.2 $v_{l'',1,f_5} \prec v_{l'',1,\alpha} \prec v_{l,1,\alpha} \prec v_{l',1,\alpha} \prec v_{l'',1,f_9}$.
This case is symmetric to the case 1.1.

2 $v_{l'',1,f_5} \prec v_{l',1,\alpha} \prec \{v_{l,1,\alpha}, v_{l'',1,\alpha}\} \prec v_{l'',1,f_9}$.
The case is symmetric to the case 1.

♠

By Claim 9, without loss of generality, we assume

$$v_{l'',1,f_5} \prec v_{l,1,\alpha} \prec v_{l',1,\alpha} \prec v_{l'',1,\alpha} \prec v_{l'',1,f_9} \tag{12}$$

and consider the routing from $w_{l'}$ to $v_{l,1,\alpha}$.

Claim 10 *The routing from $w_{l'}$ to $v_{l,1,\alpha}$ cannot pass through $(w_{l'}, v_{l',C,\alpha})$.*

Reason: We have three cases as follows.

- $l'\alpha$-chain has exactly one interval set.
  Consider the position of $v_{l',C,\alpha}$. Since $v_{l'',1,f_9} \in \mathcal{L}(w_{l'}, v_{l',C,\alpha})$, by Lemma 4.4, we have the cyclic structure $v_{l'',1,f_5} \prec v_{l,1,\alpha} \prec v_{l',C,\alpha} \prec v_{l',1,\alpha} \prec v_{l'',1,\alpha} \prec v_{l'',1,f_9} \prec v_{l'',1,f_{13}}$. Again, by Lemma 4.4, $\mathcal{L}(w_{l'}, v_{l',C,\alpha})$ cannot contain $v_{l,1,\alpha}$.

- $l'\alpha$-chain has exactly two interval sets—$X_{l',\alpha}, Y_{l',\alpha}$.
  Assume $v_{l',1,\alpha} \in X_{l',\alpha}$. Consider the position of $Y_{l',\alpha}$. Since $\mathcal{L}(w_{l'}, v_{l',C,\alpha})$ contains $v_{l'',1,f_9}$ but not $v_{l'',1,f_5}, v_{l'',1,f_{13}}$, we have the cyclic structure $v_{l'',1,f_5} \prec v_{l,1,\alpha} \prec X_{l',\alpha} \prec v_{l'',1,\alpha} \prec v_{l'',1,f_9} \prec Y_{l',\alpha} \prec v_{l'',1,f_{13}}$ by Lemma 4.5. Again, by Lemma 4.5, $\mathcal{L}(w_{l'}, v_{l',C,\alpha})$ cannot contain $v_{l,1,\alpha}$.

- $l'\alpha$-chain has exactly 3 interval sets—$X_{l',\alpha}, Y_{l',\alpha}, Z_{l',\alpha}$.
  Assume $v_{l,1,\alpha} \in \mathcal{L}(w_{l'}, v_{l',C,\alpha})$. Recall that $\mathcal{L}(w_{l'}, v_{l',C,\alpha})$ contains $v_{l,1,\alpha}$, $v_{l',1,\alpha}$, $v_{l'',1,\alpha}, v_{l'',1,f_9}$, and $v_{l',1,\alpha} \in \mathcal{L}(u_\alpha, v_{l',1,\alpha})$, and that the meaning of $X_{l,f}, Y_{l,f}, Z_{l,f}$ stated in Page 14. Then, we have $v_{l',1,\alpha} \in Y_{l',\alpha}$. Since $v_{l'',1,f_5}, v_{l'',1,f_{13}} \notin \mathcal{L}(w_{l'}, v_{l',C,\alpha})$, in other words, $\exists x_1, x_2 \in \mathcal{L}(w_{l'}, v_{l',C,\alpha})$ and $\exists y_1, y_2 \notin \mathcal{L}(w_{l'}, v_{l',C,\alpha})$ such that $y_1 \prec x_1 \prec Y_{l',\alpha} \prec x_2 \prec y_2$, which is a contradiction to Lemma 4.8, no matter where the position of $X_{l',\alpha}$ and $Z_{l',\alpha}$ are.

♠

Consider the routing from $w_{l'}$ to $v_{l,1,\alpha}$. By Claim 10, the routing cannot start with the edge $(w_{l'}, v_{l',C,\alpha})$. Also, it cannot start with the edges $(w_{l'}, v_{l',C,f_i})$, $\forall i \in [1,17]$, because of the similar reason in Claim 6. Let the routing starts with the edge $(w_{l'}, v_{l',C,\beta})$ where $\beta \neq \alpha, f_1, \ldots, f_{17}$. *i.e.* $v_{l,1,\alpha} \in \mathcal{L}(w_{l'}, v_{l',C,\beta})$. We are going to find out the possible position of $v_{l',1,\beta}$.

Claim 11 $v_{l',1,\beta} \in \langle v_{l'',1,f_1}, v_{l'',1,f_9} \rangle$.

Reason: Assume $v_{l',1,\beta} \notin \langle v_{l'',1,f_1}, v_{l'',1,f_9}\rangle$. Recall $v_{l,1,\alpha} \in \mathcal{L}(w_{l'}, v_{l',C,\beta})$ and $v_{l',1,\beta} \in \mathcal{L}(u_\beta, v_{l',1,\beta})$. Since this $l'\beta$-chain is normal, $\mathcal{L}(w_{l'}, v_{l',C,\beta}) \cup \mathcal{L}(u_\beta, v_{l',1,\beta})$ is an interval. Then, $v_{l',1,\beta} \notin \langle v_{l'',1,f_1}, v_{l'',1,f_9}\rangle$ implies that $\mathcal{L}(w_{l'}, v_{l',C,\beta}) \cup \mathcal{L}(u_\beta, v_{l',1,\beta})$ contains

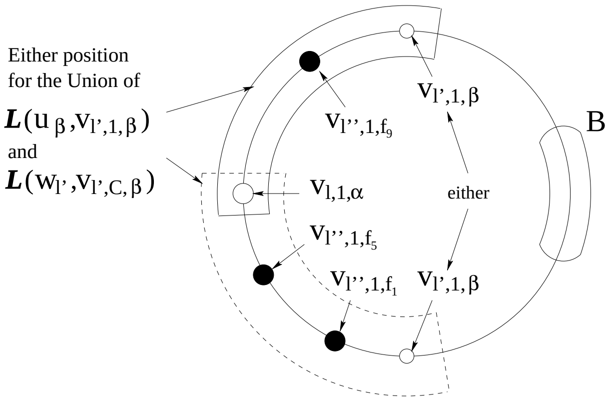

Figure 18: Two cases of $\mathcal{L}(w_{l'}, v_{l',C,\beta}) \cup \mathcal{L}(u_\beta, v_{l',1,\beta})$.

either $\{v_{l'',1,f_1}, v_{l'',1,f_5}\}$ or $v_{l'',1,f_9}$ (Figure 18).

- $\{v_{l'',1,f_1}, v_{l'',1,f_5}\} \subset \mathcal{L}(w_{l'}, v_{l',C,\beta}) \cup \mathcal{L}(u_\beta, v_{l',1,\beta})$.
  $v_{l'',1,f_1}, v_{l'',1,f_5} \notin \mathcal{L}(u_\beta, v_{l',1,\beta})$; otherwise, the routing path from u_β to $v_{l'',1,f_1}$ (to $v_{l'',1,f_5}$, too) will be no shorter than $2D-1$. Hence, $v_{l'',1,f_1}, v_{l'',1,f_5} \in \mathcal{L}(w_{l'}, v_{l',C,\beta})$. However, it is a contradiction to Claim 7 since $\mathcal{L}(w_{l'}, v_{l',C,\beta})$ will include $v_{l'',1,f_1}, \ldots, v_{l'',1,f_5}$.

- $v_{l'',1,f_9} \in \mathcal{L}(w_{l'}, v_{l',C,\beta}) \cup \mathcal{L}(u_\beta, v_{l',1,\beta})$.
  $v_{l'',1,f_9} \notin \mathcal{L}(u_\beta, v_{l',1,\beta})$; otherwise, the routing path from u_β to $v_{l',1,f_9}$ will be no shorter than $2D-1$. However, $v_{l'',1,f_9} \notin \mathcal{L}(w_{l'}, v_{l',C,\beta})$, because $v_{l'',1,f_9} \in \mathcal{L}(w_{l'}, v_{l',C,\alpha})$ and by Property 2, $\mathcal{L}(w_{l'}, v_{l',C,\beta}) \cap \mathcal{L}(w_{l'}, v_{l',C,\alpha}) = \emptyset$. Contradiction.

♠

Claim 12 $v_{l',1,\beta} \in \langle v_{l'',1,f_1}, v_{l,1,\alpha}\rangle \cup \langle v_{l'',1,\alpha}, v_{l'',1,f_9}\rangle$.

Reason: $v_{l',1,\beta} \notin \langle v_{l,1,\alpha}, v_{l'',1,\alpha}\rangle$; otherwise $v_{l',1,\beta} \in \mathcal{L}(w_l, v_{l,C,\alpha})$, implying a routing path from w_l to $v_{l',1,\beta}$ no shorter than $\frac{3}{2}D-1$, contradicting Claim 1. Combining with the Claim 11, result follows. ♠

Combining (12) with Claim 12, we have two cases:

1. $v_{l'',1,f_1} \prec v_{l',1,\beta} \prec v_{l,1,\alpha} \prec v_{l',1,\alpha} \prec v_{l'',1,\alpha} \prec v_{l'',1,f_9}$,

2. $v_{l'',1,f_1} \prec v_{l,1,\alpha} \prec v_{l',1,\alpha} \prec v_{l'',1,\alpha} \prec v_{l',1,\beta} \prec v_{l'',1,f_9}$.

Considering the routing from $w_l, w_{l''}$ to $v_{l',1,\beta}$, like (11), the above two cases become

1. $v_{l'',1,f_1} \prec \{v_{l,1,\beta}, v_{l',1,\beta}, v_{l'',1,\beta}\} \prec v_{l,1,\alpha} \prec v_{l',1,\alpha} \prec v_{l'',1,\alpha} \prec v_{l'',1,f_9}$

2. $v_{l'',1,f_1} \prec v_{l,1,\alpha} \prec v_{l',1,\alpha} \prec v_{l'',1,\alpha} \prec \{v_{l,1,\beta}, v_{l',1,\beta}, v_{l'',1,\beta}\} \prec v_{l'',1,f_9}$

By similar technique using in Claim 9, the cyclic structure inside $\{v_{l,1,\beta}, v_{l',1,\beta}, v_{l'',1,\beta}\}$ is either $v_{l,1,\beta} \prec v_{l',1,\beta} \prec v_{l'',1,\beta}$ or $v_{l'',1,\beta} \prec v_{l',1,\beta} \prec v_{l,1,\beta}$.

Claim 13 $v_{l',1,\beta} \in \langle v_{l'',1,f_1}, v_{l,1,\alpha}\rangle$.

Reason: From Claim 12, we have either $v_{l',1,\beta} \in \langle v_{l'',1,f_1}, v_{l,1,\alpha}\rangle$ or $v_{l',1,\beta} \in \langle v_{l'',1,\alpha}, v_{l'',1,f_9}\rangle$.

Assume $v_{l',1,\beta} \in \langle v_{l'',1,\alpha}, v_{l'',1,f_9}\rangle$. By (12), we have

$$v_{l'',1,f_5} \prec v_{l,1,\alpha} \prec v_{l',1,\alpha} \prec v_{l'',1,\alpha} \prec v_{l',1,\beta} \prec v_{l'',1,f_9}$$

and hence,

$$v_{l'',1,f_5} \prec v_{l,1,\alpha} \prec v_{l',1,\alpha} \prec v_{l'',1,\alpha} \prec v_{l,1,\beta} \prec v_{l',1,\beta} \prec v_{l'',1,\beta} \prec v_{l'',1,f_9},$$

where the position of $v_{l,1,\beta}$ and $v_{l'',1,\beta}$ can be exchanged; this case is left to the reader.

Recall that $v_{l,1,\alpha} \in \mathcal{L}(w_{l'}, v_{l',C,\beta})$ and $v_{l',1,\beta} \in \mathcal{L}(u_\beta, v_{l',1,\beta})$, and $l'\beta$-chain is normal. Then, $v_{l,1,\beta} \in \mathcal{L}(u_\beta, v_{l',1,\beta}) \cup \mathcal{L}(w_{l'}, v_{l',C,\beta})$ (Figure 19).

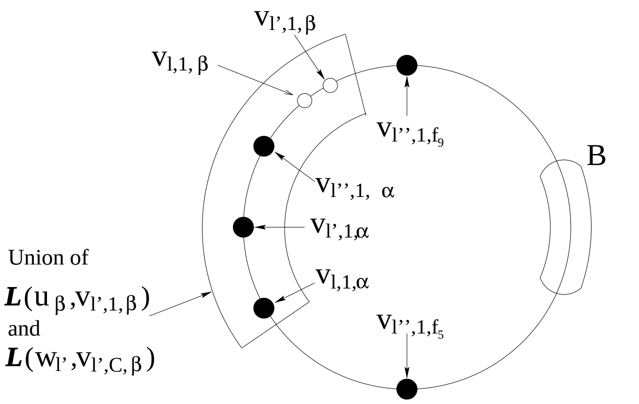

Figure 19: $v_{l,1,\beta} \in \mathcal{L}(w_{l'}, v_{l',C,\beta}) \cup \mathcal{L}(u_\beta, v_{l',1,\beta})$.

$v_{l,1,\beta} \notin \mathcal{L}(u_\beta, v_{l',1,\beta})$, since $v_{l,1,\beta} \in \mathcal{L}(u_\beta, v_{l,1,\beta})$ by the longest path assumption (2D-K). Hence, $v_{l,1,\beta} \in \mathcal{L}(w_{l'}, v_{l',C,\beta})$. However, the interval $\mathcal{L}(w_{l'}, v_{l',C,\beta})$ containing $v_{l,1,\beta}$ and $v_{l,1,\alpha}$ will contain $v_{l',1,\alpha}$ or $v_{l'',1,f_5}, v_{l'',1,f_9}$. Contradiction to Property 2 on $w_{l'}$ for both cases. ♠

By Claim 13,

$$v_{l'',1,f_1} \prec v_{l,1,\beta} \prec v_{l',1,\beta} \prec v_{l'',1,\beta} \prec v_{l,1,\alpha} \prec v_{l',1,\alpha} \prec v_{l'',1,\alpha} \prec v_{l'',1,f_9} \quad \text{or}$$
$$v_{l'',1,f_1} \prec v_{l'',1,\beta} \prec v_{l',1,\beta} \prec v_{l,1,\beta} \prec v_{l,1,\alpha} \prec v_{l',1,\alpha} \prec v_{l'',1,\alpha} \prec v_{l'',1,f_9}$$

Without loss of generality, we assume the former case and we have

$$\underbrace{v_{l'',1,f_1}}_{\in\mathcal{L}(w_l,v_{l,C,f_1})} \prec \underbrace{v_{l,1,\beta} \prec v_{l',1,\beta} \prec v_{l'',1,\beta}}_{\in\mathcal{L}(w_l,v_{l,C,\beta})} \prec \underbrace{v_{l,1,\alpha} \prec v_{l',1,\alpha} \prec v_{l'',1,\alpha}}_{\in\mathcal{L}(w_l,v_{l,C,\alpha})} \prec \underbrace{v_{l'',1,f_9}}_{\in\mathcal{L}(w_l,v_{l,C,f_9})}.$$

Now we are going to find the position of $\mathcal{L}(w_{l''}, v_{l'',C,f_5})$.

Claim 14 $v_{l'',1,f_1} \prec \mathcal{L}(w_l, v_{l,C,f_5}) \prec v_{l,1,\beta} \prec v_{l',1,\beta} \prec v_{l'',1,\beta} \prec v_{l,1,\alpha} \prec v_{l',1,\alpha} \prec v_{l'',1,\alpha} \prec v_{l'',1,f_9}$

Reason: Assume the contrary,

$$v_{l'',1,f_1} \prec \underbrace{v_{l,1,\beta} \prec v_{l',1,\beta} \prec v_{l'',1,\beta}}_{\in\mathcal{L}(w_l,v_{l,C,\beta})} \prec \mathcal{L}(w_l, v_{l,C,f_5}) \prec \underbrace{v_{l,1,\alpha} \prec v_{l',1,\alpha} \prec v_{l'',1,\alpha}}_{\in\mathcal{L}(w_l,v_{l,C,\alpha})} \prec v_{l'',1,f_9}.$$

Recall that $v_{l,C,f_5} \in \mathcal{L}(w_l, v_{l,C,f_5})$, $v_{l',1,\beta} \in \mathcal{L}(u_\beta, v_{l',1,\beta})$ and $v_{l,1,\alpha} \in \mathcal{L}(w_{l'}, v_{l',C,\beta})$. Since $l'\beta$-chain is normal, $\mathcal{L}(w_{l'}, v_{l',C,\beta}) \cup \mathcal{L}(u_\beta, v_{l',1,\beta})$ is an interval. Since, under the assumption of longest path $(2D-1)$, $\mathcal{L}(w_{l'}, v_{l',C,\beta})$ and $\mathcal{L}(u_\beta, v_{l',1,\beta})$ cannot both contain $v_{l'',1,f_1}$ and $v_{l'',1,f_9}$, $\mathcal{L}(w_{l'}, v_{l',C,\beta}) \cup \mathcal{L}(u_\beta, v_{l',1,\beta})$ must contain $\langle v_{l',1,\beta}, v_{l,1,\alpha}\rangle$. Then,

$$v_{l'',1,f_1} \prec v_{l,1,\beta} \prec \underbrace{v_{l',1,\beta} \prec v_{l'',1,\beta} \prec v_{l,C,f_5} \prec v_{l,1,\alpha}}_{\in\mathcal{L}(w_{l'},v_{l',C,\beta})\cup\mathcal{L}(u_\beta,v_{l',1,\beta})} \prec v_{l',1,\alpha} \prec v_{l'',1,\alpha} \prec v_{l'',1,f_9}.$$

$v_{l,C,f_5} \notin \mathcal{L}(u_\beta, v_{l',1,\beta})$; otherwise, $v_{l'',1,\beta} \in \mathcal{L}(u_\beta, v_{l',1,\beta})$, which implies a routing path from u_β to $v_{l'',1,\beta}$ no shorter than $2D-1$. Hence, $v_{l,C,f_5} \in \mathcal{L}(w_{l'}, v_{l',C,\beta})$.

Consider the routing from $w_{l'}$ to v_{l,C,f_5}. In order to avoid a path longer than $2D-1$, the routing path must be $w_{l'}, u_\beta, w_l, v_{l,C,f_5}$. Hence, $v_{l,C,f_5} \in \mathcal{L}(u_\beta, v_{l,1,\beta})$.

Obviously, $v_{l,1,\beta} \in \mathcal{L}(u_\beta, v_{l,1,\beta})$. $v_{l',1,\beta}, v_{l'',1,\beta} \notin \mathcal{L}(u_\beta, v_{l,1,\beta})$ and $v_{l,1,\beta}, v_{l,C,f_5} \in \mathcal{L}(u_\beta, v_{l,1,\beta})$ imply that $\mathcal{L}(u_\beta, v_{l,1,\beta})$ will contain B. *i.e.*

$$\underbrace{a_1 \prec v_{l'',1,f_1} \prec v_{l,1,\beta}}_{\in\mathcal{L}(u_\beta,v_{l,1,\beta})} \prec \underbrace{v_{l',1,\beta} \prec v_{l'',1,\beta}}_{\notin\mathcal{L}(u_\beta,v_{l,1,\beta})} \prec$$
$$\underbrace{v_{l,C,f_5} \prec \{v_{l,1,\alpha}, v_{l',1,\alpha}, v_{l'',1,\alpha}\} \prec v_{l'',1,f_9} \prec a_2 \prec B \prec a_1}_{\in\mathcal{L}(u_\beta,v_{l,1,\beta})}.$$

Pick a element in B, say $v_{l_b,1,f_5}$. In order to avoid a routing path longer than $2D-1$, the routing path must be $u_\beta, w_l, u_{f_5}, v_{l_b,1,f_5}$. This routing path implies that

$v_{l_b,1,f_5} \in \mathcal{L}(w_{l,C,f_5})$. But, $v_{l'',1,f_5} \in \mathcal{L}(w_{l,C,f_5})$. Therefore, either $v_{l'',1,f_1}$ or $v_{l'',1,f_9}$ will be contained in $\mathcal{L}(w_{l,C,f_5})$. A contradiction to Property 2 on w_l exists because $v_{l'',1,f_1} \in \mathcal{L}(w_l, v_{l,1,f_1})$ and $v_{l'',1,f_9} \in \mathcal{L}(w_l, v_{l,1,f_9})$ by Claim 5. ♠

Then, by Claim 14 and $v_{l'',1,f_5} \in \mathcal{L}(w_l, v_{l,C,f_5})$, we have the cyclic structure

$$v_{l'',1,f_5} \prec \underbrace{\{v_{l,1,\beta}, v_{l',1,\beta}, v_{l'',1,\beta}\}}_{\beta\text{th flap}} \prec \underbrace{\{v_{l,1,\alpha}, v_{l',1,\alpha}, v_{l'',1,\alpha}\}}_{\alpha\text{th flap}} \prec v_{l'',1,f_9}. \tag{13}$$

For convenience, we use a short-hand notation S_i for $\{v_{l,1,i}, v_{l',1,i}, v_{l'',1,i}\}$. Like the steps from (11) to (13), if we consider the routing from $w_{l'}$ to $v_{l,C,\beta}$ or to $v_{l'',C,\beta}$ whichever nearer to $v_{l'',1,f_5}$, we will have another γ such that

$$v_{l'',1,f_5} \prec \underbrace{S_\gamma}_{\gamma\text{th flap}} \prec \underbrace{S_\beta}_{\beta\text{th flap}} \prec \underbrace{S_\alpha}_{\alpha\text{th flap}} \prec v_{l'',1,f_9}.$$

Inductively, we must have infinite many number of S_{λ_i}, $i \in [1, \mathcal{F}]$, $F < \mathcal{F}$, such that

$$v_{l'',1,f_5} \prec \cdots\cdots \prec S_{\lambda_2} \prec S_{\lambda_1} \prec S_\gamma \prec S_\beta \prec S_\alpha \prec v_{l'',1,f_9}$$

However, we have only F flaps. It is a contradiction to the graph structure. Then, the proof of Main Lemma 5.1 is completed.

6 The $2D - K$ Lower Bound, $K \geq 3$

Theorem 6.1 *$\forall G_{L,C,F}$, where $L \geq 15$, $C \geq 3$ and $F \geq \{[(2(L-2)(20(3\lfloor\frac{D}{K}\rfloor + 15) + 4) + 3)(L-1) + 1](L-1) + (\lfloor\frac{D}{K}\rfloor + 2)L + 4\}L$, there is no labeling scheme in which the longest path is shorter than $2D - K$, $K \geq 3$.*

Proof: Assume there exists a labeling scheme such that the longest path is shorter than $2D - K$.

By the definition of $G_{L,C,F}$, we have L layers and F flaps. There are LF elements in the set $\{v_{l,1,f} | l \in [1, L], f \in [1, F]\}$, referred to as R. Consider the routing from u_F. By Property 1, $\cup_{l=1}^{l=L}\mathcal{L}(u_F, v_{l,1,F})$ will contain R. Since R is distributed in the L edges' interval labels from u_F and $|R| = LF$, by the Pigeon Hole Principle, there exists an edge, say $(u_F, v_{L,1,F})$, such that $\mathcal{L}(u_F, v_{L,1,F})$ will contain at least F elements of R. Let Q be the set of these elements, and therefore $(u_F, v_{L,1,F})$ contain Q, $Q \subset R$. Then, $|Q| \geq F \geq \{[(2(L-2)(20(3\lfloor\frac{D}{K}\rfloor + 15) + 4) + 3)(L-1) + 1](L-1) + (\lfloor\frac{D}{K}\rfloor + 2)L + 4\}L$ by the choice of F in the theorem statement.

Partition Q into $R_1 \cup R_2 \cup \ldots \cup R_L$ where $R_l = \{v_{l,1,f} |$ for some $f \in [1, F]\} \subset Q$. Again by the Pigeon Hole Principle, there exists an l_a such that $|R_{l_a}| \geq \{[(2(L-$

$2)(20(3\lfloor\frac{D}{K}\rfloor+15)+4)+3)(L-1)+1](L-1)+(\lfloor\frac{D}{K}\rfloor+2)L+4\}$. Let $S_{l_a} = R_{l_a} - \{v_{l_a,1,F}\}$ and certainly, $\mathcal{L}(u_F, v_{L,1,F})$ contains S_{l_a}. Let $p = |S_{l_a}| \geq \{[(2(L-2)(20(3\lfloor\frac{D}{K}\rfloor+15)+4)+3)(L-1)+1](L-1)+(\lfloor\frac{D}{K}\rfloor+2)L+3\}$. Without loss of generality, assume $S_{l_a} = \{v_{l_a,1,f} | f \in [1,p]\}$ and $v_{l_a,1,p} \prec v_{l_a,1,1} \prec v_{l_a,1,2} \prec \cdots \prec v_{l_a,1,p-2} \prec v_{l_a,1,p-1}$. Under the assumption on the longest path, the routing from u_F to the elements of S_{l_a} should pass through an Lth layer, where L can be, but not necessarily, equal to l_a. So, we have $v_{l_a,1,i} \in \mathcal{L}(w_L, v_{L,C,i})$, $\forall i \in [1,p]$; otherwise, the routing path from u_F to $v_{l_a,1,i}$, $i \in [1,p]$, is no shorter than $2D-1$. Then, by Property 2 on w_L, we have

$$v_{l_a,1,p} \prec \mathcal{L}(w_L, v_{L,C,1}) \prec \mathcal{L}(w_L, v_{L,C,2}) \prec \cdots \prec \mathcal{L}(w_L, v_{L,C,p-2}) \prec v_{l_a,1,p-1}.$$

Hence, $\mathcal{L}(u_F, v_{L,1,F})$ contains $\mathcal{L}(w_L, v_{L,C,1}), \ldots, \mathcal{L}(w_L, v_{L,C,p-2})$, and in other words, $\mathcal{L}(u_F, v_{L,1,F})$ contains $v_{L,C,1}, \ldots, v_{L,C,p-2}$. The length of routing paths from w_L through $v_{L,C,f} \in \langle v_{l_a,1,p}, v_{l_a,1,p-1}\rangle$ must be less than $\frac{3}{2}D-3$, where f is not necessarily in $[1,p]$; otherwise, the length of a routing path from u_F through that edge $(w_L, v_{L,C,f})$ will be greater than $2D-3$.

Consider the main Lemma 5.1. Define interval

$$A = \langle v_{l_a,1,p}, v_{l_a,1,p-1}\rangle - \{v_{l_a,1,p}, v_{l_a,1,p-1}\},$$

which is a subinterval of $\mathcal{L}(u_F, v_{L,1,F})$. For each $l_b \in [1,L]$, $l_b \neq L$, $l_b \neq l_a$, if there are $(2(L-2)(20(3\lfloor\frac{D}{K}\rfloor+15)+4)+3)(L-1)+2$ elements out of the set $\{v_{l_b,1,f} | f \in [1,p-2]\}$ which is not in A, then we can make B the interval containing the $(2(L-2)(20(3\lfloor\frac{D}{K}\rfloor+15)+4)+3)(L-1)+2$ elements of the set $\{v_{l_b,1,f} | f \in [1,p-2]\}$ and by Lemma 5.1, $\exists$ a routing path from $(w_L, v_{L,C,f})$, $v_{L,C,f} \in A$, which is no shorter than $\frac{3}{2}D-1$.

Therefore, there are at most $(2(L-2)(20(3\lfloor\frac{D}{K}\rfloor+15)+4)+3)(L-1)+1$ elements of the set $\{v_{l_b,1,f} | f \in [1,p-2]\}$ which are not in A. Hence, there are at most $[(2(L-2)(20(3\lfloor\frac{D}{K}\rfloor+15)+4)+3)(L-1)+1](L-1)$ elements in the set $\{v_{l_b,1,f} | l_b \in [1,L], l_b \neq L, l_b \neq l_a, f \in [1,p-2]\}$ which are not in A. These elements belong to at most $[(2(L-2)(20(3\lfloor\frac{D}{K}\rfloor+15)+4)+3)(L-1)+1](L-1)$ flaps. In other words, there are $(\lfloor\frac{D}{K}\rfloor+2)L+1$ flaps, say, 1st, ..., $(\lfloor\frac{D}{K}\rfloor+2)L+1$th flaps, such that the elements in $\{v_{l,1,f} | l \in [1,L], f \in [1,(\lfloor\frac{D}{K}\rfloor+2)L+1]\}$ belong to A and belong to $\mathcal{L}(u_F, v_{L,1,F})$.

Among the chains in the first $(\lfloor\frac{D}{K}\rfloor+2)L+1$ flaps, by Lemma 4.2, there are at most $\lfloor\frac{D}{K}\rfloor+2$ abnormal chains in a layer. There are L layers and at most $(\lfloor\frac{D}{K}\rfloor+2)L$ flaps containing these abnormal chains. Hence, we have one flap, say the 1st flap, whose elements, (*ie.* $v_{l,1,1} | l \in [1,L]$) belong to A, such that $\forall l \in [1,L]$, $l1$-chain is normal.

Consider the 1st flap. Referring to the routing from u_F to any element in $\{v_{l,1,1} | l \in [1,L]\}$, $\{v_{l,1,1} | l \in [1,L]\} \subset \mathcal{L}(w_L, v_{L,C,1})$; otherwise, one of the routing paths from u_F will be no shorter than $2D-1$.

If no interval containing $\{v_{l,1,1}|l \in [1,L]\}$ is disjoint with $\{v_{l,1,f}|l \in [1,L], f \in [2,F]\} - \{v_{L,1,F}\}$, then say $\mathcal{L}(w_L, v_{L,C,1})$ contains a $v_{l',1,f}$, $f \neq 1$. The routing path from u_F to $v_{l',1,f}$ is $u_F, w_L, u_1, w_{l''}, v_{l',1,f}$, $l'' \neq L$, which is no shorter than $2D-1$. Hence, there exists an interval T containing $\{v_{l,1,1}|l \in [1,L]\}$, but not any elements in $\{v_{l,1,f}|l \in [1,L], f \in [2,F]\} - \{v_{L,1,F}\}$ and $T \subseteq \mathcal{L}(w_L, v_{L,C,1})$. We call this as *disjoint property of* T.

Consider T. There are two marginal elements, say t_1, t_2 in $\{v_{l,1,1}|l \in [1,L]\}$. Without loss of generality, assume that no marginal elements of T are in the first $L-2$ flaps and assume

$$t_1 \prec \{v_{l,1,1}|l \in [1,L-2]\} \prec t_2 \prec t_1.$$

Moreover, by Lemma 4.11, there are at most two elements in $\{v_{l,C,1}|l \in [1,L-2]\}$ which are not in $\langle t_1,t_2\rangle$. Without loss of generality, assume that these two elements are in the $L-3$th and $L-2$th flaps. So, our scope is restricted on the set $\{v_{l,1,1}, v_{l,C,1}|l \in [1,L-4]\} \subset \langle t_1,t_2\rangle$. Let us look into the interval $\langle t_1,t_2\rangle$. Without loss of generality, assume

$$t_1 \prec v_{1,1,1} \prec v_{2,1,1} \prec \cdots \prec v_{L-4,1,1} \prec t_2. \tag{14}$$

By Property 2 on u_1, and recall $t_1, t_2 \in \{v_{L-1,1,1}, v_{L,1,1}\}$, we have

$$t_1 \prec \mathcal{L}(u_1, v_{1,1,1}) \prec \mathcal{L}(u_1, v_{2,1,1}) \prec \cdots \prec \mathcal{L}(u_1, v_{L-4,1,1}) \prec t_2. \tag{15}$$

For $l \in [1, L-4]$, $v_{l,C,1} \in \langle t_1,t_2\rangle \subset T \subset \mathcal{L}(w_L, v_{L,C,1}) \subset \mathcal{L}(u_F, v_{L,1,F})$. If $v_{l,C,1} \notin \mathcal{L}(u_1, v_{l,1,1})$, the routing path from u_F to $v_{l,C,1}$ will be $u_F, w_L, u_1, w_{l'}, u_f, w_l, v_{l,C,1}$, which is no shorter than $2D-1$. Therefore, $v_{l,C,1} \in \mathcal{L}(u_1, v_{l,1,1})$. By (15),

$$t_1 \prec \{v_{1,1,1}, v_{1,C,1}\} \prec \{v_{2,1,1}, v_{2,C,1}\} \prec \cdots \prec \{v_{L-4,1,1}, v_{L-4,C,1}\} \prec t_2. \tag{16}$$

For $l \in [1, L-4]$, if $v_{l,C,f} \in \langle t_1,t_2\rangle$, $f \neq 1$, then $v_{l,C,f} \in \mathcal{L}(u_1, v_{l,1,1})$; otherwise, by similar argument as above, the routing path from u_F to $v_{l,C,f}$ is no shorter than $2D-1$.

If $\forall f \in [2,F]$, $v_{6,C,f} \in \langle t_1,t_2\rangle$, then $\forall f \in [1,F]$, $v_{6,C,f} \in \mathcal{L}(u_1, v_{6,1,1})$. By Lemma 4.2, there are $F - \lfloor \frac{D}{K} \rfloor - 2$ $v_{6,C,f}$'s belonging to $\mathcal{L}(u_1, v_{6,1,1})$, and each $v_{6,C,f}$ belongs to a normal chain. By Lemma 4.6 and 4.10, there are $F - \lfloor \frac{D}{K} \rfloor - 6$ pairs of $v_{6,1,f}, v_{6,C,f} \in \mathcal{L}(u_1, v_{6,1,1}) \subset T$, contradicting the disjoint property of T. Therefore, there exists at least one $f' \in [2,F]$ such that $v_{6,C,f'} \notin \langle t_1,t_2\rangle$.

Consider the routing from $v_{6,C-1,1}$. $\mathcal{L}(v_{6,C-1,1}, v_{6,C,1})$ will contain $v_{6,C,f}$, $\forall f \in [1,F]$; otherwise, the routing path from $v_{6,C-1,1}$ to $v_{6,C,f}$, for some $f \in [1,F]$, will be no shorter than $2D-3$.

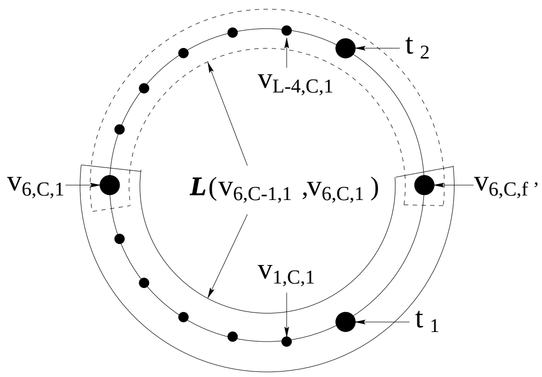

Figure 20: Two choices for $\mathcal{L}(v_{6,C-1,1}, v_{6,C,1})$.

As shown in Figure 20, $\mathcal{L}(v_{6,C-1,1}, v_{6,C,1})$ will contain $v_{1,C,1}, v_{2,C,1}, v_{3,C,1}, v_{4,C,1}, v_{5,C,1}$ or $v_{7,C,1}, v_{8,C,1}, \ldots, v_{L-4,C,1}$. Since $L \geq 15$, each choice (Figure 20) has at least five elements.

We assume that $\mathcal{L}(v_{6,C-1,1}, v_{6,C,1})$ contains $v_{1,C,1}, v_{2,C,1}, \ldots, v_{5,C,1}$ and the other case is just similar. The routing path from $v_{6,C-1,1}$ to $v_{5,C,1}$ should be $v_{6,C-1,1}, w_6, u_f$, $w_5, v_{5,C,1}$, $(f \neq 1)$; otherwise, the path will be no shorter than $2D$. Hence, $\mathcal{L}(w_6, v_{6,C,f})$ and $\mathcal{L}(u_f, v_{5,1,f})$ contain $v_{5,C,1}$. Consider the routing from $v_{6,C-1,1}$ to $v_{3,C,1}$. Then, $\exists f'' \neq 1$ such that $\mathcal{L}(w_6, v_{6,C,f''})$ and $\mathcal{L}(u_{f''}, v_{3,1,f''})$ contain $v_{3,C,1}$.

We have two cases.

- $f = f''$.

  That means $v_{3,C,1}, v_{5,C,1} \in \mathcal{L}(w_6, v_{6,C,f})$.

  If $v_{4,C,1} \notin \mathcal{L}(w_6, v_{6,C,f})$, $v_{4,C,1} \in \mathcal{L}(w_6, v_{6,C,f^o})$, $f^o \neq f$. Then, by (16),

  $$t_1 \prec \underbrace{v_{3,C,1}}_{\in\mathcal{L}(w_6,v_{6,C,f})} \prec \underbrace{v_{4,C,1}}_{\in\mathcal{L}(w_6,v_{6,C,f^o})} \prec \underbrace{v_{5,C,1}}_{\in\mathcal{L}(w_6,v_{6,C,f})} \prec \{v_{6,1,1}, v_{6,C,1}\} \prec t_2.$$

  Then,

  $$t_1 \prec \underbrace{v_{3,C,1}}_{\in\mathcal{L}(u_1,v_{3,1,1})} \prec \{v_{4,C,1}, v_{6,C,f^o}\} \prec \underbrace{v_{5,C,1}}_{\in\mathcal{L}(u_1,v_{5,1,1})} \prec \underbrace{\{v_{6,1,1}, v_{6,C,1}\}}_{\in\mathcal{L}(u_1,v_{6,1,1})} \prec t_2.$$

  Contradiction to the fact $v_{6,C,f^o} \in \langle t_1, t_2\rangle \Rightarrow v_{6,C,f^o} \in \mathcal{L}(u_1, v_{6,1,1})$. Hence, $v_{4,C,1} \in \mathcal{L}(w_6, v_{6,C,f})$, implying $v_{4,C,1} \in \mathcal{L}(u_f, v_{4,1,f})$. Then, we have

  $$t_1 \prec \underbrace{v_{3,C,1}}_{\in\mathcal{L}(u_f,v_{3,1,f})} \prec \underbrace{v_{4,C,1}}_{\in\mathcal{L}(u_f,v_{4,1,f})} \prec \underbrace{v_{5,C,1}}_{\in\mathcal{L}(u_f,v_{5,1,f})} \prec t_2.$$

  In other words,

  $$t_1 \prec v_{3,C,1} \prec \{v_{4,C,1}, v_{4,1,f}\} \prec v_{5,C,1} \prec t_2.$$

Therefore, $v_{4,1,f} \in \langle t_1, t_2 \rangle \subset T$, contradicting to the disjoint property of T.

- $f \neq f''$.
  We have two subcases. Firstly, $v_{6,C,f''} \in \langle t_1, t_2 \rangle$, which implies that $v_{6,C,f''}$ is "around" $v_{6,C,1}$ because $v_{6,C,f''}, v_{6,C,1} \in \mathcal{L}(u_1, v_{6,1,1})$. We show the possible interval for $\mathcal{L}(w_6, v_{6,C,f''})$ in Figure 21(a). Second, $v_{6,C,f''} \notin \langle t_1, t_2 \rangle$, the possible

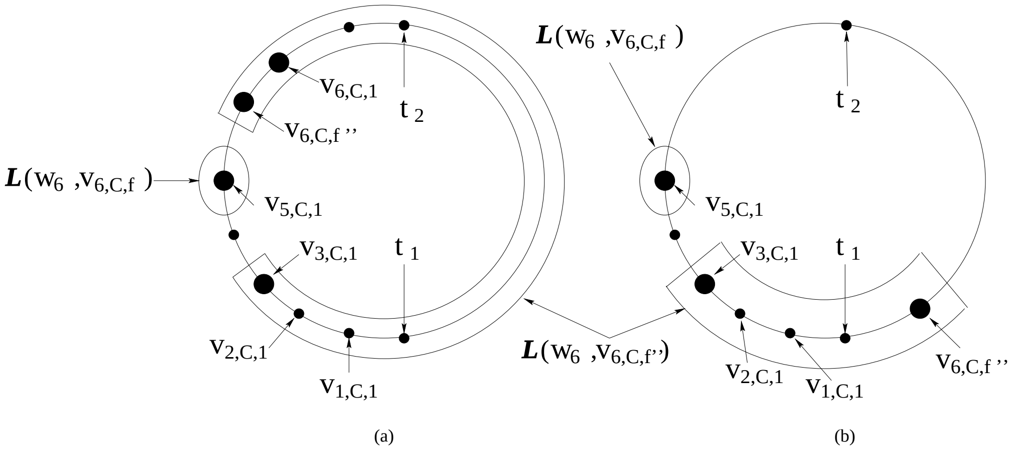

Figure 21: Two possible cases for $\mathcal{L}(w_6, v_{6,C,f''})$.

  interval for $\mathcal{L}(w_6, v_{6,C,f''})$ is as shown in Figure 21(b).

  Referring to Figure 21, the choice is limited because the existence of $\mathcal{L}(w_6, v_{6,C,f})$ which is disjoint with $\mathcal{L}(w_6, v_{6,C,f''})$. In both cases,

  $$v_{1,C,1}, v_{2,C,1}, v_{3,C,1} \in \mathcal{L}(w_6, v_{6,C,f''}),$$

  implying that $v_{1,C,1} \in \mathcal{L}(u_{f''}, v_{1,1,f''})$, $v_{2,C,1} \in \mathcal{L}(u_{f''}, v_{2,1,f''})$ and $v_{3,C,1} \in \mathcal{L}(u_{f''}, v_{3,1,f''})$. Then,

  $$t_1 \prec \underbrace{v_{1,C,1}}_{\in \mathcal{L}(u_{f''}, v_{1,1,f''})} \prec \underbrace{v_{2,C,1}}_{\in \mathcal{L}(u_{f''}, v_{2,1,f''})} \prec \underbrace{v_{3,C,1}}_{\in \mathcal{L}(u_{f''}, v_{3,1,f''})} \prec t_2,$$

  implying

  $$t_1 \prec v_{1,C,1} \prec \{v_{2,C,1}, v_{2,1,f''}\} \prec v_{3,C,1} \prec t_2.$$

  Therefore, $v_{2,1,f''} \in \langle t_1, t_2 \rangle \subset T$, contradicting the disjoint property of T.

Both cases are not valid, and this completes the proof of Theorem 6.1. $\diamondsuit$

7 Main Results

Theorem 7.1 *There exists a graph such that no labeling scheme can have the longest path shorter than $2D - o(D)$, where $D = O(n)$.*

Proof: By Theorem 6.1, $2D-K$ is the lower bound. If $2D-\Omega(D)$ is an upper bound on the longest path for $G_{L,C,F}$, by the definition of Ω, $\exists\delta > 0$ such that $(2-\delta)D$ is the upper bound. However, by substituting $K = \frac{\delta}{2}D$ into Theorem 6.1, $(2-\frac{\delta}{2})D$ is a lower bound. Contradiction follows. $\diamondsuit$

Theorem 7.2 *There exists a graph such that no labeling scheme can have the longest path shorter than $2D-3$ where $D = O(\sqrt{n})$.*

Proof: By substituting $K = 3$ into Theorem 6.1, the result follows. $\diamondsuit$

8 Open Problems

- Is there a better lower bound for 1-label interval routing?
- Are there algorithms for an upper bound of smaller than $2D$?
- Are there any other types of graphs yielding a lower bound of $2D - O(1)$?
- The lower bound $2D-3$ is deduced from a graph with n not less than $1,491,345,315$. Are there any graphs with a smaller order yielding a lower bound of $2D - O(1)$?

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